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MRC Technical Summary Report #1957

ON SYSTEMS OF TODA TYPE

W. Symes

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May 1979

(Received March 9, 1979)

Approved for public release
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Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

and

National Science Foundation
Washington, D. C. 20550

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

This report presents a group-theoretical interpretation of the structure of certain completely integrable Hamiltonian systems. These systems generalize the nonperiodic Toda Lattice system. Since all of them owe their special properties to the same group-theoretical mechanism, we call them systems of Toda type. This mechanism also provides a foundation for understanding many other completely integrable systems, including the Korteweg-de Vries equation, which will not be discussed here. We give an invariant definition of the class of systems of Toda type, and a method for explicit integration of all systems in this class in terms of rational combinations of linear exponential functions.

AMS (MOS) Subject classification: 35A25

Key Words: Toda lattice, integrals in involution, inverse spectral problem, representation theory of semisimple Lie algebras.

Work Unit Number 1 - Applied Analysis.

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024. This material is based upon work supported by the National Science Foundation under Grant No. MCS78-09525.

SIGNIFICANCE AND EXPLANATION

It is very unusual to be able to give explicit solutions of nonlinear dynamical systems in terms of well-known "elementary" functions. When this is possible, some special underlying algebraic structure of the dynamical system is usually responsible. Often a continuous (Lie) symmetry group appears in such problems, and its role is worth understanding, for application to other problems which might bear a similar relation to group theory. In this report we describe how certain constructions in the representation theory of Lie groups enable one to solve a number of differential equations, including the well-known Toda Lattice equations, in terms of rational functions of exponentials. Some of our results also illuminate the behaviour of a number of other integrable dynamical systems, among them the Korteweg-de Vries equation and the Euler equations for motion of a rigid body.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON SYSTEMS OF TODA TYPE

W. Symes

§1. Introduction

This report presents a group-theoretical interpretation of the structure of certain completely integrable Hamiltonian systems. These systems ([1], [2]) generalize the nonperiodic Toda lattice system. Since all of them owe their special properties to the same group-theoretical mechanism (Theorem 2.2 below), we call them systems of Toda type. This mechanism also provides a foundation for understanding many other completely integrable systems, including the Korteweg-de Vries equation, which will not be discussed here. We give an intrinsic definition of the class of systems of Toda type, and a method for explicit integration of all systems in this class in terms of rational combinations of linear exponential functions.

Our results amplify and extend in various directions the work of Moser [2], Adler [3], [4], Kostant [5], and others.

The study of completely integrable systems proceeds in three stages: (i) identification of the symplectic structure which gives the system its Hamiltonian character; (ii) identification of first integrals (action variables or constants of motion); (iii) identification of a complementary set of variables, and computation of their evolution under the various Hamiltonian flows associated to the first integrals, if possible in terms of "elementary" functions. This program was the stuff of early mechanics, and has long since been abandoned, for very good reasons, as an approach to the study of general Hamiltonian dynamics. The systems considered here, however, admit this approach because of their basic connection with certain Lie groups, which provides a remarkable amount of structure not otherwise to be expected. In fact, each of the above steps, as carried out below, is the reflection of group-theoretic circumstances.

(i) The symplectic manifolds on which the systems are defined are orbits of the coadjoint action of a Lie group G on the dual of its Lie algebra, with the natural symplectic structure.

This observation, which plays a crucial role, is due independently to Mark Adler and Bertram Kostant.

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(ii) The constants of motion are coadjoint-invariant polynomials of a larger Lie group L in which the Lie group G of (i) is embedded as a subgroup; their involution is equivalent to the existence of a complementary subgroup H (Theorem 2.2).

This result is due independently to Kostant, and generalizes a theorem of Adler [3]. Indeed, the work reported here began as an attempt to reconcile Adler's results with some puzzling computations in [6].

Points (i) and (ii) are explained in section 2.

(iii) On certain orbits, the coadjoint action of G coincides, along the common level sets of the integrals (ii), with an action defined by the inclusion $G \subset L$. Under a natural nondegeneracy hypthesis, these turn out to be exactly the orbits on which the L-invariant polynomials provide Lagrangian polarizations, i.e., completely integrable systems (Theorems 4.3, 4.5). This result is explained in §4, and generalizes a well known fact about Jacobi (symmetric tridiagonal) matrices, which are in a natural way a coadjoint orbit of the group of lower triangular matrices: any two Jacobi matrices (with discrete exceptions) which have the same simple spectrum, are similar by a lower triangular matrix.

(iv) The class of systems of Toda type is defined as follows. The Lie algebra $\underline{\ell}$ of L is a normal real form of a complex semi simple Lie algebra $\underline{m}$. The Lie algebra $\underline{h}$ of H is the $+1$ eigenspace of a Cartan involution of $\underline{\ell}$, i.e., the intersection of a compact real form of $\underline{m}$ with $\underline{\ell}$. The Lie algebra $\underline{g}$ of G is the complementary (to $\underline{h}$) Borel subalgebra in $\underline{\ell}$. The dual $\underline{g}^*$ is identified with $\underline{h}^\perp$ in the natural way (via the Killing form). Let $\underline{c} \subset \underline{h}^\perp$ be the corresponding Cartan subalgebra. A coadjoint orbit $\mathcal{O}$ of G in $\underline{h}^\perp$ is called a Toda orbit if (i) it contains a regular point $a \notin \underline{c}$ (this condition is the guise assumed in this setting by the nondegeneracy condition mentioned in (iii) above); (ii) $\dim \mathcal{O} = 2 \operatorname{rank} \underline{\ell}$. A system of Toda type is a Hamiltonian system on a Toda orbit $\mathcal{O}$ in $\underline{h}^\perp$, with the restriction of an L-invariant function to $\mathcal{O}$ as Hamiltonian.

In §5 a set of coordinates (not quite canonical) is constructed on each Toda orbit. Half of these are L-invariant functions, which generalize the eigenvalues of Jacobi matrices. The other half are analogous to the norming constants of Sturm-Liouville Theory.

In §6 the representation theory of $\underline{\ell}$ is used to describe the trajectories of systems of Toda type. A scheme is given based on the results of section 4, which enables the matrix elements of representers of trajectories in each irreducible representation of $\underline{\ell}$ to be expressed as rational functions of linear exponentials in "time" with coefficients rational in the matrix elements and spectrum of the initial representer.

For this last stunt, we use a Euclidean structure on each irreducible representation space, for which the Cartan involution on $\underline{\ell}$ goes over into metric adjoint. In an appendix to §5 we give a completely algebraic proof, which seems to be new, of the (well-known) existence of such a metric, by way of the Verma module construction.

We mention several ways in which the theory developed here falls short of what has been achieved by others in specific examples. First, the results of Section 6 amount to a partial solution, in the style of Gel'fand-Levitan [7], of a brand of inverse spectral problem. We fail, however, to give an a priori characterization of the set of spectral data for all representations and all Toda orbits. Also we are able to parameterize only a Zariski-open subset of the representers of points in Toda orbit having the same spectrum as a fixed point. These omissions contrast with the satisfactorily complete solutions of the inverse spectral problem for Jacobi matrices (which form a Toda orbit), described by Moser [2], Golub-Welsh [8], Golub-de Boor [9], and Case-Kac [10], on which the present theory is modeled. Kostant [5] has also given a much more complete treatment of the inverse spectral problem for the Jacobi sets in split semisimple real Lie algebras. His work includes the treatment of Moser [2] as a special case (Moser's work in turn seems to go back to work of Stietjes). Despite these failings, however, our results suffice to solve explicitly all of the systems of Toda type as described above.

Also, we fail to explicate the mapping between the coordinates of §5 and the matrix elements of various representers. It will be clear to the reader that information is needed about the orbit under the subgroup H of a maximal vector for each irreducible representation of $\underline{\ell}$.

This matter is related to the problem of characterizing spectral data, mentioned above.

The Jacobi matrices (symmetric tridiagonal matrices with positive off-diagonal entries) form a Toda orbit (this is explained, for instance, in [3]). Their immediate generalizations are the Jacobi sets in split semisimple Lie algebras (see [5]), which are the orbits of certain principal nilpotent elements. The Jacobi sets are the appropriate phase spaces for the generalized Toda lattice systems first described by Bogoyavlensky [24]. These are not, however, the only Toda orbits.

The author has described a 6-dimensional non-Jacobi Toda orbit of the group of lower triangular 4×4 matrices of determinant 1, and computed Darboux coordinates and a family of 3 Poisson-commuting functions ([25]). The author has also located a number of other Toda orbits in the $n \times n$ lower triangular group ([26]); they appear to be quite numerous. The Jacobi orbit appears to be the only such Toda orbit consisting entirely of regular elements, however, although we have not proved this.

It is interesting to note that the basic result on systems in involution (Theorem 2.2 below) has a range of applicability far beyond the class of systems of Toda type, as defined here. Some finite-dimensional examples, including a very interesting one due to Mumford, van Moerbeke, and Adler, are described in §3, with detailed matrix computation of various Hamiltonian vector fields. With slight modifications, this result also yields the various systems in involution associated with the Korteweg-de Vries equation, the "nonlinear Schrödinger" equation, the Euler-Arnold generalized rigid body motion, and the equations of geodesic flow on an ellipsoid; see [3], [11], [12]. Adler and van Moerbeke [11] use this theorem in the context of Kac-Moody Lie algebras, in their work on periodic generalized Toda lattices. It also yields the constants of motion for the sine-Gordon [13] and Kac-Moerbeke [14] equations though with the wrong symplectic structure.

ACKNOWLEDGMENT

I take great pleasure in thanking Mark Adler for many interesting conversations, Joel Robbin for help in formulating the basic result (Theorem 2.2), and Charles Conley for the opportunity to present some of this work in his seminar at the University of Wisconsin-Madison.

§2. A Theorem on Systems in Involution

A Poisson Structure is a pair $(M, \{\ \})$, M a smooth manifold and $\{\ \}$ a map

$$\{\ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

satisfying:

(i) $\{\ \}$ is antisymmetric and bilinear

(ii) $\{\ \}$ is a derivation in each argument

(iii) $\{F\{G,H\}\} + \{H\{FG\}\} + \{G\{H,F\}\} = 0$ for all triples $F, G, H \in C^\infty(M)$.

Any $H \in C^\infty(M)$ defines a vector D_H field via the prescription

$$D_H F = \{F,H\}$$

D_H is the Hamiltonian vector field of the (Hamiltonian) function H.

Symplectic structure provides a special instance of Poisson structure. Suppose M comes equipped with a closed nondegenerate two form ω. For any $H \in C^\infty(M)$ specify the vector field D_H by

$$\omega \lrcorner D_H = dH$$

and define

$$\{F,H\}_\omega = \omega(D_F \wedge D_H) \quad .$$

Then $\{,\}_\omega$ is a Poisson bracket, and

$$D_H F = \{F,H\}_\omega \quad .$$

Poisson structures form a larger class than symplectic structures. Still it is not hard to convince oneself that any Poisson structure is stratified by symplectic substructures; see some remarks in [3].

The following class of Poisson structures, studied initially by Kirillov [15], will be central in what follows. Its relevance in this context was observed independently by Adler [3] and Kostant [5].

Let G be a connected Lie group over $\mathbb{R}$, $\underline{g}$ its Lie algebra, $\underline{g}^*$ its dual. To every $x \in \underline{g}$ is associated the endomorphism $\operatorname{ad} x$ of $\underline{g}$. To every $X \in G$ is associated the

automorphism Ad X of $\underline{g}$:

$$\mathrm{Ad}\, X : y \mapsto dL_X \circ dR_X(y)$$

where $y \in \underline{g}$ is interpreted as a left-invariant vector field on G and L_X, R_X denote respectively left and right multiplication by X in G.

Denote by ad', Ad' the contragredients to ad, Ad, that is

$$\mathrm{ad}'x = -(\mathrm{ad}\, x)^*$$
$$\mathrm{Ad}'X = (\mathrm{Ad}\, X^{-1})^* \quad .$$

Thus for $\alpha \in \underline{g}^*$, $x, y \in \underline{g}$

$$\langle \mathrm{ad}'x(\alpha), y\rangle = \langle \alpha, [y,x]\rangle \quad .$$

The representations ad', Ad' of $\underline{g}$, G in End ($\underline{g}^*$) are called coadjoint.

For a smooth function $F : \underline{g}^* \to \mathbb{R}$, define the gradient $\nabla F : \underline{g}^* \to \underline{g}$ by

$$\langle \dot{\alpha}, \nabla F(\alpha)\rangle = dF(\alpha,\dot{\alpha}), \ (\alpha,\dot{\alpha}) \in \underline{g}^* \times \underline{g}^* \cong T\underline{g}^* \quad .$$

<u>Kirillov's</u> <u>Poisson</u> <u>bracket</u> on $\underline{g}^*$ is now defined by

$$\{F,G\}(\alpha) \equiv \langle \alpha[\nabla F(\alpha), \nabla G(\alpha)]\rangle$$

for $F,G \in C^\infty(\underline{g}^*)$. The proof that $(\underline{g}^*, \{\ \})$ is a Poisson structure can be found in [16], Ch. 2, as can the following facts:

The Hamiltonian vector field for $H \in C^\infty(\underline{g}^*)$ is

$$\mathcal{D}_H(\alpha) = (\alpha, \mathrm{ad}'(\nabla H(\alpha))(\alpha)) \quad .$$

Obviously $\mathcal{D}_H$ is tangent to the orbit $\mathcal{O}_\alpha^G = \mathcal{O}_\alpha$ of G through α (under Ad'). These orbits are symplectic manifolds: for

$$U_\nu = (\alpha, \mathrm{ad}'x_\nu(\alpha)) \in T_\alpha \mathcal{O}_\alpha \equiv V_\alpha \quad , \quad \nu = 1,2$$

define

$$\omega_\alpha(U_1 \wedge U_2) \equiv \langle \alpha, [x_1,x_2]\rangle \quad .$$

Then ω_α is a symplectic form on $\mathcal{O}_\alpha$.

So the stratification mentioned above is actually effected by the $\mathrm{Ad}'G$ - orbit decomposition of $\underline{g}^*$, in this case.

Note that each orbit $\mathcal{O}_\alpha$ is a homogeneous symplectic manifold, i.e. the action of G is symplectic and transitive. Note also that the restriction of the Hamiltonian vector field $\mathcal{D}_H$ to $\mathcal{O}_\alpha$ is determined already by the restriction of the Hamiltonian H to $\mathcal{O}_\alpha$.

Now suppose that $\underline{g}$ is a subalgebra of another real Lie algebra $\underline{\ell}$, with connected group L. Let $\underline{h} \subset \underline{\ell}$ be a complementary subspace. Then the dual of the inclusion $\underline{g} \subset \underline{\ell}$ is a linear isomorphism when restricted to $\underline{h}^\perp$. Use this isomorphism to identify $\underline{g}^*$ with $\underline{h}^\perp$. Of course $\underline{g}^\perp \sim \underline{h}^*$ also, but only one of these identifications will be of much use. Denote by $\Pi_{\underline{g}}, \Pi_{\underline{h}}, \Pi_{\underline{g}^\perp}, \Pi_{\underline{h}^\perp}$ the projections relative to the decompositions $\underline{\ell} = \underline{g} \oplus \underline{h}$, $\underline{\ell}^* = \underline{g}^\perp \oplus \underline{h}^\perp$.

Lemma 2.1.

(i) For $F \in C^\infty(\underline{\ell}^*)$, denote also by F the restriction to $\underline{h}^\perp \simeq \underline{g}^*$. Denote by $\nabla_{\underline{g}}F : \underline{g}^* \to \underline{g}$ the gradient of F as a function on $\underline{g}^*$, and by $\nabla F : \underline{\ell}^* \to \underline{\ell}$ the gradient as a function on $\underline{h}^\perp \subset \underline{\ell}$. Then for $\alpha \in \underline{h}^\perp$,

$$\nabla_{\underline{g}}F(\alpha) = \Pi_{\underline{g}}\nabla F(\alpha) \quad .$$

(ii) Denote by $\mathrm{ad}'_{\underline{g}}$ the representation of $\underline{g}$ in $\mathrm{End}(\underline{h}^\perp)$ induced by the coadjoint representation of $\underline{g}$ on $\underline{g}^*$ by the identification $\underline{g}^* \simeq \underline{h}^\perp$. Then for $x \in \underline{g}$, $\alpha \in \underline{h}^\perp \subset \underline{\ell}^*$

$$\mathrm{ad}'_{\underline{g}}x(\alpha) = \Pi_{\underline{h}^\perp}\mathrm{ad}'x(\alpha)$$

where on the r.h.s. ad' means $\mathrm{ad}' : \underline{\ell} \to \mathrm{End}(\underline{\ell}^*)$.

The proof is trivial. The notations introduced in the statement will be used throughout.

You compute easily that $\mathrm{ad}'_{\underline{g}}$ induces a Poisson bracket on $\underline{h}^\perp$ given by

$$\{F_1,F_2\}(\alpha) = \langle\alpha, [\Pi_{\underline{g}}\nabla F_1(\alpha), \Pi_{\underline{g}}\nabla F_2(\alpha)]\rangle$$

and Hamiltonian vector fields

$$\mathcal{D}_F(\alpha) = (\alpha, \Pi_{\underline{h}^\perp}\mathrm{ad}'(\Pi_{\underline{g}}\nabla F(\alpha))\ (\alpha)) \quad .$$

Now introduce the class $\mathcal{S} \subset C^\infty(\underline{\ell}^*)$ of ad'-invariant functions on $\underline{\ell}^*$, defined by

$$\mathcal{S} = \{F \in C^\infty(\underline{\ell}^*) : \langle\nabla F(\alpha), \mathrm{ad}'x(\alpha)\rangle = 0 \text{ for all } \alpha \in \underline{\ell}^*, x \in \underline{\ell}\} .$$

These are, of course, exactly those functions constant on the Ad'-orbits of L in $\underline{\ell}^*$.

The following Theorem is basic to what follows:

Theorem 2.2. Suppose that $\underline{h}$ is also a subalgebra of $\underline{\ell}$. Then:

(i) $\mathcal{S}|_{\underline{h}^\perp}$ is a system in involution, i.e. for $F_1, F_2 \in \mathcal{S}|_{\underline{h}^\perp}$, $\{F_1, F_2\} \equiv 0$.

(ii) For $F \in \mathcal{S}|_{\underline{h}^\perp}$, the Hamiltonian vector field is given by

$$\begin{aligned} D_H(\alpha) &= (\alpha, \mathrm{ad}'(\Pi_{\underline{g}} \nabla F(\alpha))\,(\alpha)) \\ &= (\alpha, -\mathrm{ad}'(\Pi_{\underline{h}} \nabla F(\alpha))\,(\alpha)) \quad . \end{aligned}$$

Remarks

1) In his paper [5], B. Kostant has given a version of this theorem. Its proof is a generalization of Adler's proof of Theorem 1 in Ref. [3].

2) The formulae (ii), especially the second, generalize the formula for infinitesimal isospectral deformation of matrices or operators, the utility of which is the context of the KdV equation, Toda lattice, etc., was pointed out by P. Lax in [17]. The thing to notice is that the projection $\Pi_{\underline{h}^\perp}$ is missing in front of ad' in both right-hand sides.

Proof.

(i) For $\alpha \in \underline{h}^\perp$, $x \in \underline{\ell}$, and $F \in \mathcal{S}$, we have

$$\begin{aligned} 0 &= \langle \mathrm{ad}'x(\alpha), \nabla F(\alpha)\rangle \\ &= \langle \alpha, [\nabla F(\alpha), x]\rangle = \langle -\mathrm{ad}'(\nabla F(\alpha))\,(\alpha), x\rangle \end{aligned}$$

i.e. $F \in \mathcal{S}$ if and only if

$$\begin{aligned} 0 &= \mathrm{ad}'(\nabla F(\alpha))\,(\alpha) \\ &= \mathrm{ad}'(\Pi_{\underline{g}} \nabla F(\alpha))\,(\alpha) + \mathrm{ad}'(\Pi_{\underline{h}} \nabla F(\alpha))\,(\alpha) \quad . \end{aligned}$$

So for $F_1, F_2 \in \mathcal{S}$, $\alpha \in \underline{h}^\perp$,

$$
\begin{aligned}
\{F_1,F_2\}(\alpha) &= \langle \alpha, [\Pi_{\underline{g}}\nabla F_1(\alpha), \Pi_{\underline{g}}\nabla F_2(\alpha)]\rangle \\
&= \langle \mathrm{ad}'(\Pi_{\underline{g}}\nabla F_2(\alpha))\,(\alpha), \Pi_{\underline{g}}\nabla F_1(\alpha)\rangle \\
&= -\langle \mathrm{ad}'(\Pi_{\underline{h}}\nabla F_2(\alpha))\,(\alpha), \Pi_{\underline{g}}\nabla F_1(\alpha)\rangle \\
&= -\langle \alpha, [\Pi_{\underline{g}}\nabla F(\alpha), \Pi_{\underline{h}}\nabla F_2(\alpha)]\rangle \\
&= \langle \alpha\ [\Pi_{\underline{h}}\nabla F_2(\alpha), \Pi_{\underline{g}}\nabla F_1(\alpha)]\rangle \\
&= \langle \mathrm{ad}'(\Pi_{\underline{g}}\nabla F_1(\alpha))\,(\alpha), \Pi_{\underline{h}}\nabla F_2(\alpha)\rangle \\
&= -\langle \mathrm{ad}'(\Pi_{\underline{h}}\nabla F_1(\alpha))\,(\alpha), \Pi_{\underline{h}}\nabla F_2(\alpha)\rangle \\
&= -\langle \alpha, [\Pi_{\underline{h}}\nabla F_2(\alpha), \Pi_{\underline{h}}\nabla F_1(\alpha)]\rangle \\
&= 0 \ .
\end{aligned}
$$

Note that the last step works only if $\underline{h}$ is a subalgebra of $\underline{\ell}$.

(ii) For $F \in S$, $\alpha \in \underline{h}^{\perp}$,

$$
\begin{aligned}
D_H(\alpha) &= \langle \alpha, \Pi_{\underline{h}^{\perp}}\mathrm{ad}'(\Pi_{\underline{g}}\nabla F(\alpha))\,(\alpha)\rangle \\
&= \langle \alpha, -\mathrm{ad}'(\Pi_{\underline{h}}\nabla F(\alpha))\,(\alpha)\rangle
\end{aligned}
$$

where the $\Pi_{\underline{h}^{\perp}}$ becomes superfluous because $\mathrm{ad}'\underline{h}(\underline{h}^{\perp}) \subset \underline{h}^{\perp}$, as can easily be checked. However, because of the ad'-invariance of F, this is the same as

$$
= \langle \alpha, \mathrm{ad}(\Pi_{\underline{g}}\nabla F(\alpha))\,(\alpha)\rangle \ .
$$

q.e.d.

Since the proof of Theorem 2.1 depends only on the properties of the gradients $\nabla F(\alpha)$, $F \in S$, we have the following local version:

Cor. 2.3. Let $S(U)$ denote the collection of locally ad-invariant functions in the open set $U \subset \underline{\ell}^*$, i.e. those $F \in C^{\infty}(U)$ satisfying

$$
\mathrm{ad}'(\nabla F(\alpha))\,(\alpha) = 0, \ \alpha \in U \ .
$$

Then $S(U)\big|_{\underline{h}^{\perp}\cap U}$ is a system in involution on $U \cap \underline{h}^{\perp}$, and formulae (ii) above hold at all $\alpha \in U \cap \underline{h}^{\perp}$.

§3. Examples

All of the instances of Theorem 2.2 investigated to date involve the compact/Borel decompositions of semisimple algebras, at least so far as this author knows. This section is devoted to a description of several such decompositions $\underline{\ell} = \underline{g} \oplus \underline{h}$, and a display of exemplary computations for each resulting system in involution.

The computations get rather explicit in part II. This explicitness is meant on the one hand to connect our invariantly-formulated results with many matrix manipulations found in other papers on the subject. On the other hand, a collection of apparent computational coincidences suggests the need for a systematic study of the behaviour of the systems of Theorem 2.2 under various algebraic and functorial manipulations of the underlying Lie algebras. Such a program remains to be carried out.

The decompositions are presented in part I, in terms of a root system for the split semisimple real Lie algebra $\underline{\ell}$.

The Cartan-Killing form of $\underline{\ell}$ allows a canonical identification $\underline{\ell} \cong \underline{\ell}^*$, under which the adjoint and coadjoint representations coincide. Explicit formulae are given for the Hamiltonian vector fields and Poisson bracket, and $\underline{h}^{\perp}$ is determined.

The first three examples are variations on the compact/Borel decomposition theme. The first is the real compact/Borel decomposition of a normal real form (see [18], Ch. 1 for the definition), and forms the subject of the last two articles of this paper. It was first considered in this context by Mark Adler and Bertram Kostant. Examples 4 and 5 are due to Ken Macrae and the author. Example 6 was invented by Mark Adler [11] to treat the asymmetric Toda Lattice systems of van Moerbeke and Mumford [19]. Our contribution is to show how Adler's result follows from Theorem 2.2. Example 7 is particularly noteworthy: it shows that Theorem 2.2 may actually have no punch in certain situations, i.e. the collection of commuting vector fields may collapse to the zero field.

In part II $\underline{\ell}$ becomes uniformly $s\ell(n,\mathbb{R})$, and explicit matrix computations of all quantities mentioned in Theorem 2.2 are given.

I. Some decompositions of semisimple Lie algebras

1. Let $\underline{m}$ be a complex semisimple Lie algebra, and select a Cartan subalgebra $\underline{j} \subset \underline{m}$. Let Δ be the root system for the pair $(\underline{m},\underline{j})$, P a system of positive roots,

$$\underline{m} = \underline{j} \oplus \sum_{\gamma \in P} \underline{m}_{\gamma} \oplus \underline{m}_{-\gamma}$$

the root space decomposition. According to [20], Ch. 4, there is a Weyl basis of $\underline{m}$: that is, there are basis vectors Z_{γ} of $\underline{m}_{\gamma}$ such that

$$[Z_{\gamma},Z_{\beta}] = c_{\gamma,\beta}Z_{\gamma+\beta} \quad , \quad \gamma \neq -\beta$$

where $c_{\gamma,\beta} = 0$ unless $\alpha + \beta$ is a non-zero root and $c_{\gamma,\beta} = -c_{-\gamma,-\beta}$. Also

$$[Z_{\gamma},Z_{-\gamma}] = e_{\gamma} \in \underline{j}$$

$$[e_{\gamma},Z_{\beta}] = K(\gamma,\beta)\ Z_{\beta}$$

where $K(,)$ denotes the Killing form, viewed also as a (nondegenerate) form on $\underline{j}^*$ in the obvious way.

Denote by $\underline{\ell}$ the $\mathbb{R}$-linear span of $\{e_{\gamma},Z_{\gamma} : \gamma \in \Delta\}$. Let σ be the linear map $\underline{\ell} \to \underline{\ell}$ defined by

$$\sigma(e_{\gamma}) = e_{\gamma},\ \sigma(Z_{\gamma}) = Z_{-\gamma} \quad .$$

Facts: (see especially [18] Ch. 1)

1) $\underline{\ell}$ is a semisimple Lie algebra over $\mathbb{R}$. $\underline{\ell}$ is called a normal real form of $\underline{m}$, thanks to 5) below (see [18]).

2) σ is an anti-automorphism of $\underline{\ell}$, and $\sigma^2 = 1$.

3) $\underline{h} = \{w \in \underline{\ell} : \sigma(w) = -w\}$ is a subalgebra of $\underline{\ell}$, and K restricted to $\underline{h}$ is negative-definite.

4) $\underline{h}^{\perp} = \{w \in \ell : K(w,x) = 0\ \forall x \in \underline{h}\}$
$= \{w \in \underline{\ell} : \sigma(w) = w\}$.

5) $\underline{c} = \mathbb{R}$-linear span of e_{α}, $\alpha \in \Delta$, is a Cartan subalgebra of $\underline{\ell}$, and $\underline{c} \subset \underline{h}^{\perp}$.

6) $[\underline{h}^\perp, \underline{h}^\perp] \subset \underline{h}$, and K is positive-definite when restricted to $\underline{h}^\perp$, which consists of semi-simple elements.

7) Write $\underline{n}_\pm = \bigoplus_{\gamma \in P} \mathbb{R} \cdot Z_{\pm\gamma}$. Then $\underline{n}_\pm$ are nilpotent subalgebras of $\underline{\ell}$ and $\underline{g} = \underline{c} \oplus \underline{n}_-$ is a subalgebra of $\underline{\ell}$ complementary to $\underline{h}$. In fact, $\underline{g}$ is a Borel subalgebra - i.e. a maximal solvable subalgebra of $\underline{\ell}$.

Since K is nondegenerate, you may use it to identify $\underline{\ell}$ with $\underline{\ell}^*$. Having done this, you compute, using the antisymmetry of ad with respect to K,

$$\mathrm{ad}' = \mathrm{ad} \quad .$$

When $\underline{g}^*$ is identified with $\underline{h}^\perp$ as before, for $F \in \mathcal{S}$ (which is now the class of ad-invariant functions on $\underline{\ell}$ i.e. the class functions), $\alpha \in \underline{h}^\perp$

$$\begin{aligned} D_F(\alpha) &= [\alpha, \Pi_{\underline{h}} \nabla F(\alpha)] \\ &= [\Pi_{\underline{g}} \nabla F(\alpha), \alpha] \quad . \end{aligned}$$

The Poisson bracket is $(\alpha \in \underline{h}, F_1, F_2 \in C^\infty(\underline{\ell}^*)) : \{F_1, F_2\}(\alpha) = K(\alpha, [\Pi_{\underline{g}} \nabla F_1(\alpha), \Pi_{\underline{g}} \nabla F_2(\alpha)])$. The number of independent differentials of functions in $\mathcal{S}$ at regular points is equal to the rank r of $\underline{\ell}$ (which is the same as the rank of $\underline{m}$).

2. Complex Borel decomposition

The complex Lie algebra $\underline{m} = \underline{\ell} \otimes_{\mathbb{R}} \mathbb{C}$ may also be decomposed over $\mathbb{C}$. The subalgebra $\underline{g}_{\mathbb{C}}$ is now a maximal solvable complex subalgebra:

$$\underline{g}_{\mathbb{C}} = \underline{j} \oplus \sum_{\alpha \in P} \mathbb{C} \cdot (Z_{-\alpha})$$

and its complement is the (-1) eigenspace of σ, extended complex-linearly:

$$h_{\mathbb{C}} = \sum_{\alpha \in P} \mathbb{C} \cdot (Z_\alpha - Z_{-\alpha}) \quad .$$

The expressions for Hamiltonian vector fields and Poisson brackets remain as in Example 1.

The class functions on $\underline{m}$ are holomorphic, and yield r = (rank $\underline{m}$) $\mathbb{C}$-linearly independent holomorphic differentials at each regular $\alpha \in \underline{m}$.

3. Hermitian Borel Decomposition

The complex Lie algebra $\underline{m}$ may also be considered as a real Lie algebra which leads to the decomposition $\underline{m} = \tilde{\underline{g}} \oplus \tilde{\underline{h}}$

$$\tilde{\underline{g}} = \underline{c} \oplus \sum_{\gamma\in P} \mathbb{C} \cdot z_{-\gamma}$$

$$\tilde{\underline{h}} = \sqrt{-1} \cdot \underline{c} \oplus \underline{h} \oplus \sqrt{-1} \cdot \underline{h}^{\perp} \quad .$$

Here $\tilde{\underline{g}}$ is a maximal solvable real subalgebra of $\underline{m}$, and $\tilde{\underline{h}}$ is a compact real form of $\underline{m}$.

Again the Hamiltonian vector fields and Poisson brackets are given by the expressions in Example 1. Since the underlying action is the same as in Example 2, the class functions are still holomorphic, hence there are r linearly independent (over $\mathbb{R}$) differentials at each regular $\alpha \in \underline{m}$ (Cauchy-Riemann equations).

The following two examples are based on $s\ell(n,\mathbb{R})$. The standard choice of CSA is $\underline{c} = d_0(n,\mathbb{R})$, the $n \times n$ diagonal matrices with zero trace. Then $\underline{n}_{\pm} = n_{\pm}(n,\mathbb{R})$, the strictly $\left\{\begin{smallmatrix}\text{upper}\\ \text{lower}\end{smallmatrix}\right\}$ triangular matrices. The involution σ turns into ordinary matrix transposition, denoted by superscript ^{t}. Also convenient are

$$t_{\pm}(n,\mathbb{R}) = d_0(n,\mathbb{R}) \oplus n_{\pm}(n,\mathbb{R}) \quad .$$

4. Let $\underline{\ell} = s\ell(2n,\mathbb{R})$, and select $\underline{g}$ and $\underline{h}$ according to

$$\underline{g}_{\mathbb{H}} = \left\{\left(\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right) : A \in t_-(n,\mathbb{R}),\ B,\ C,\ D \in n_-(n,\mathbb{R})\right\}$$

$$\underline{h}_{\mathbb{H}} = sp(n,\mathbb{R}) = \left\{\left(\begin{array}{c|c} A & B \\ \hline C & -A^t\end{array}\right) : B = B^t,\ C = C^t\right\} \quad .$$

In these formulae the matrix blocks are $n \times n$. An easy computation shows that

$$\underline{h}^{\perp}_{\mathbb{H}} = \left\{\left(\begin{array}{c|c} A & B \\ \hline C & A^t\end{array}\right) : B^+ = -B,\ C^+ = -C\right\} \quad .$$

At each regular $\alpha \in \underline{h}^{\perp}$ there are n independent differentials of class functions on $\underline{\ell}$.

5. Let $\underline{\ell} = s\ell(n,\mathbb{R})$,

$$\underline{g} = t_-(n,\mathbb{R}) \ , \ \underline{h} = n_+(n,\mathbb{R}) \quad ,$$

$$\underline{h}^\perp = t_+(n,\mathbb{R}) \quad .$$

Note that the class functions for $\alpha \in \underline{h}$ depend only on the diagonal entries.

6. In this example, $\underline{\ell}$ is as in example 1.

Let $\underline{p} = \underline{\ell} \oplus \underline{\ell}$, and make $\underline{p}$ into a Lie algebra according to the prescription

$$[(x,y), (z,w)] = ([x,z], -[y,w]) \quad .$$

Then $\underline{p}$ is semisimple with Killing form

$$\hat{K}((x,y), (z,w)) = K(x,z) - K(y,w) \quad .$$

Denote by h the antidiagonal subalgebra:

$$\underline{h} = \{(x,-x) : x \in \ell\} \quad .$$

Then $\underline{h}^\perp$, identified with a subset of $\underline{p}$, is just $\underline{h}$ itself: $h^\perp = \underline{h}$.

Set

$$\underline{g} = \{(x+z, z+y) : x \in \underline{n}_-, y \in \underline{n}_+, z \in \underline{c}\} \quad .$$

Then $\underline{g}$ is a subalgebra of $\underline{p}$ - in fact, $[\underline{g},\underline{g}] \subset \underline{n}_- \oplus \underline{n}_+$. It will be verified in part II below, in the course of computing projections, that

$$\underline{p} = \underline{g} \oplus \underline{h} \quad .$$

<u>Claim</u>: The class $S|_{\underline{h}}$ of restrictions of class functions on p to $\underline{h}^\perp = \underline{h}$, coincides with the class $S_{\overline{\Delta}}$ of functions on h of the form

$$F(x,-x) = f(x)$$

where f is a class function on $\underline{\ell}$.

To see this denote by $\overline{\Delta}$ the antidiagonal map : $x \mapsto (x,-x)$, and compute the differential of $f = F \circ \overline{\Delta}$, for any $F \in C^\infty(\underline{p})$:

$$df(x,\dot{x}) = dF \circ d\bar{\Delta}(\dot{x},\dot{x})$$
$$= dF((x,-x),\ (\dot{x},-\dot{x}))\ .$$

In particular, if $\dot{x} = [x,u]$ this is

$$df(x,\ [x,u]) = dF((x,-x),\ ([x,u],\ -[x,u]))$$
$$= dF((x,-x),\ [(x,-x),\ (u,-u)])\ .$$

So, if F is a class function on $\underline{p}$, then f is a class function on $\underline{\ell}$, and coincides with the restriction of F to $\underline{h}$ as above.

Conversely, suppose f is a class funtion on $\underline{\ell}$. A function of $\underline{p}$ whose restriction to $\underline{h}$ may be identified with f as above is, for instance

$$F(x,y) \quad f(x)\ .$$

Then

$$dF((x,-x),\ [(x,-x),\ (n,v)])$$
$$= dF((x,-x),\ ([x,u],\ [x,v]))$$
$$= df(x,\ [x,u]) = 0$$

since f is a class function on $\underline{\ell}$. So any class function on $\underline{\ell}$ is the restriction of a class function on $\underline{p}$ to $\underline{h}$, and the claim is proved.

Note that the pairing via $\hat{K}$ between $\underline{g}$ and $\underline{h}$ is

$$\hat{K}((\xi,\eta),\ (x,-x)) = K(\xi+\eta,\ x)\ .$$

Theorem 2.2 now implies Theorem 2 of Adler and van Moerbeke [11]: the ad-invariant functions on $\underline{\ell}$, transferred to the antidiagonal $\underline{h} = \underline{h}^{\perp} \subset \underline{p}$ as above, form a system in involution.

7. Same as Example 1 with $\underline{\ell} = s\ell(n,\mathbb{R})$, but reverse the roles of $\underline{g}$ and $\underline{h}$.

II. Computations for $s\ell(n)$

In examples 1, 2, 3, and 5, make the following choices:

$$\underline{\ell} = s\ell(n,\mathbb{R}) \qquad m = s\ell(n,\mathbb{C})$$
$$\underline{j} = d_0(n,\mathbb{C}) \qquad \underline{c} = d_0(n,\mathbb{R})$$
$$\underline{n}_+ = n_+(n,\mathbb{R})\ .$$

The basic decomposition is $\underline{\ell} = \underline{n}_- + \underline{n}_+ + \underline{c}$. A useful notation for the projections relative to this decomposition is

$$A = A_+ + A_0 + A_-$$

$$A_\pm \in \underline{n}_\pm \, , \; A_0 \in \underline{c} \; .$$

Matrix transpose realizes the antiautomorphism σ, and is denoted by superscript ^{t}.

The rest of this section is a list of useful projections and Hamiltonian vector fields. The Killing form is, up to an irrelevant constant,

$$K(A,B) = \mathrm{tr}\, AB \; .$$

In each example, the class S is generated by functions of the form $F_K = \frac{1}{K} \mathrm{tr}\, A^K$. The most interesting (or most ubiquitous) such Hamiltonian is $F_2 = \frac{1}{2} \mathrm{tr}\, A^2$. The gradient is easily computed:

$$\nabla F_2(A) = A \; .$$

The Hamiltonian vector field for F_2 is written out explicitly in each case: it is

$$\dot{A} = [A, \Pi_{\underline{h}} A] \; .$$

1. $\Pi_{\underline{h}} A = A_+ - A_+^t$

   $\Pi_{\underline{g}} A = A_- + A_+^t + A_0$

   $\underline{h}^\perp = \{A_- + A_0 + A_-^t\}$.

   For $A \in \underline{h}$, $\Pi_{\underline{g}} A = 2A_- + A_0$, and the H.V.F. for F_2 is

$$\dot{A} = [A_- + A_0 + A_-^t, A_-^t - A_-]$$

$$= 2[A_-, A_-^t] + [A_0 A_-^t] - [A_0, A_-] \; .$$

   The restriction of this V.F. to the Jacobi orbit of $G = \exp \underline{g}$ in $\underline{h}^\perp$ (to which the V.F. if tangent, of course) yields Flaschka's version of the Toda Lattic equations. The matrix

formulae above were first written by Flaschka [21], and were used by Adler to prove the involution of $\mathcal{S}$ in this case [3]. See also van Moerbeke [22].

2. The formulae in this case are the same as in example 1, except that each matrix is now complex. In terms of the real and imaginary parts, $A = A' + \sqrt{-1}\, A''$, the H.V.F. For F_2 is

$$\dot{A}' = T(A',A') - T(A'',A'')$$

$$\dot{A}'' = T(A',A'') + T(A'',A')$$

where,

$$T(A,B) := [A, \Pi_{\underline{h}} B] \quad .$$

3. $$\tilde{\underline{g}} = \{A'_- + iA''_- + A'_0 + \quad\} \qquad (i = \sqrt{-1})$$

$$\tilde{\underline{h}} = \{A'_+ - A'^{\,t}_+ + i(A''_- + A''_0 + A''^{\,t}_-)\} \qquad (= \underline{u}(\ ))$$

$$\tilde{\underline{h}}^\perp = \{A'_- + A'_0 + A'^{\,t}_- + i(A''^{\,t}_- - A''_-)\} \; .$$

For $A \in \tilde{\underline{h}}^\perp$, $A = A' + iA''$

$$\Pi_{\tilde{\underline{h}}} A = A'^{\,t}_- - A'_- + i(A''_- + A''^{\,t}_-) \quad .$$

So the H.V.F. for F_2 is

$$\dot{A}' = [A', \Pi_{\underline{h}} A'] - [A'', \Pi_{\underline{h}^\perp} A'']$$

$$= T(A',A') + T(\Pi_{\underline{h}^\perp} A'', \Pi_{\underline{h}^\perp} A'')$$

$$\dot{A}'' = [A', \Pi_{\underline{h}^\perp} A''] + [\Pi_{\underline{h}} A', A''] \quad .$$

4. For $\left(\begin{array}{c|c} A & B \\ \hline C & A^+ \end{array}\right) \in \underline{h}^{\perp}_{\mathbb{H}}$,

$$\Pi_{\underline{h}_{\mathbb{H}}} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} X & Y \\ \hline Z & -X^+ \end{array}\right)$$

with $X = -A_- - A_0 + A_+$, $Y = B_- + B_-^+$, $Z = C_- + C_-^+$.

The H.V.F. for F_2 becomes

$$\dot{A} = [A,X] + BZ - YC$$

$$\dot{B} = AY - BX^+ - XB + YA^+$$

$$\dot{C} = CX + A^+Z - ZA + X^+C \quad .$$

Notice that $B = C = 0$ is an invariant submanifold for this flow, on which the vector field reduces to

$$\dot{A} = [A,X] = 2[A_- + A_0, A_+]$$

which is an asymmetric version of the Toda lattice system - compare examples 1, 6.

5. $\underline{g} = \{A_- + A_0\}$ $\underline{h} = \{A_+\}$

$\underline{h}^{\perp} = \{A_0 + A_+\}$

for $A \in \underline{h}^{\perp}$, $\Pi_{\underline{h}} A = A_+$, so the H.V.F. for F_2 is

$$\dot{A}_0 = 0 \quad , \quad \dot{A}_+ = [A_0, A_+] \quad .$$

Notice that this system is linear, and that A_0 is fixed under <u>all</u> the flows F_K.

6. The projections on $\underline{g}$ and $\underline{h}$ are

$$\Pi_{\underline{g}}(A,B) = (A_- + B_- + \tfrac{1}{2}(A_0+B_0), \tfrac{1}{2}(A_0+B_0) + B_+ + A_+)$$

$$\Pi_{\underline{h}}(A,B) = (-B_- + \tfrac{1}{2}(A_0-B_0) + A_+, B_- + \tfrac{1}{2}(B_0-A_0) - A_+)$$

which shows that $\underline{p} = \underline{g} \oplus \underline{h}$ as asserted in part I. To carry out the prescription of Theorem 2.2, you need an Ad-invariant function on $\underline{\ell} \oplus \underline{\ell}$ which reduces to F_2 on the diagonal subspace, as per part I. Since any such extension will do, choose

$$\bar{F}_2(A,B) \equiv \frac{1}{2} \operatorname{tr} A^2 \quad .$$

Then

$$\nabla\bar{F}_2(A,B) = (A,0)$$

$$\Pi_{\underline{h}}\nabla\bar{F}_2(A,-A) = (\frac{1}{2} A_0 + A_+, -\frac{1}{2} A_0 - A_+)$$

and the H.V.F. for $\bar{F}_2$ is $(\dot{A},\dot{A})$ with

$$\dot{A} = [A, \frac{1}{2} A_0 + A_+]$$

which are the non-symmetric Toda equations of Mumford - van Moerbeke [19] and Adler - van Moerbeke [11].

In general, for F a class function on $s\ell(n,R)$ and $\bar{F}(x,y) \equiv F(x)$, you obtain

$$\dot{A} = [A, \frac{1}{2}(\nabla F(A))_0 + (\nabla F(A))_+]$$

(compare [11]).

7. Notice that with $\underline{g} = \underline{o}(n,\mathbb{R})$, $\underline{h} = t_-(n,\mathbb{R})$, you have $\underline{h}^{\perp} = n_-(n,\mathbb{R})$. Since all the ad-invariant functions on $s\ell(n,\mathbb{R})$ vanish identically on n_-, Theorem 2.2 yields exactly nothing in the way of nontrivial commuting vector fields in this case, the problem being that the isotropy algebra $\underline{\ell}_{\alpha}$ for $\alpha \in \underline{h}^{\perp}$ is actually contained in $\underline{h}$. Much more is made of this idea in the next section.

§4. Completely Polarized Orbits

This section is devoted to the proof of a remarkable fact about those orbits of $\underline{g}$ on which the $\underline{\ell}$ - ad'-invariant class S defines a complete polarization. This property is a generalization of the following observation from linear algebra:

Any two nearby symmetric Jacobi matrices with the same spectrum are conjugate by a lower triangular matrix.

As might be suspected, this fact is behind the Gel'fand-Levitan - style solution of the inverse spectral problem on certain orbits, to be presented in §6.

In this section $\underline{\ell}$ is an arbitrary Lie algebra over $\mathbb{R}$ with connected group L. In particular, $\underline{\ell}$ is not required to be semisimple. As in §2, $\underline{\ell}$ is decomposed as a direct sum of subalgebras $\underline{g}$ and $\underline{h}$.

The central object is the class S of ad'-invariant functions on $\underline{\ell}^*$. The main facts about this class are contained in the next lemma. In it, the ad'-isotropy algebra of $\alpha \in \underline{c}^*$ is denoted $\underline{\ell}_\alpha$: i.e.

$$\underline{\ell}_\alpha = \{w \in \underline{\ell} : ad'w(\alpha) = 0\} \quad .$$

<u>Lemma 4.1</u>

a) For some open set $R \subset \underline{\ell}^*$, $\alpha \in R$ implies

$$\dim_{\underline{\ell}_\alpha} = d \equiv \inf_{\beta \in \underline{\ell}^*} \dim \underline{\ell}_\beta \quad .$$

The elements of R are called co-regular. R is a Zariski - open set of the affine space $\underline{\ell}^*$.

b) For $\alpha \in R$ there is a neighborhood $U \subset \underline{\ell}^*$ of α so that

$$\underline{\ell}_\alpha = \{\nabla F(\alpha) : F \in S(U)\}$$

and

$$O_\alpha^L \cap U = \{\beta \in \underline{\ell}^* : F(\beta) = F(\alpha) \forall F \in S(U)\} \quad .$$

In particular, $(\alpha, \dot{\alpha})$ is tangent to O_α^L if and only if $\langle \dot{\alpha}, \nabla F(\alpha) \rangle = 0$ for all $F \in S(U)$.

The proofs are elementary.

Obviously situations like that of example 7, §3, are uninteresting: *there are no non-trivial orbits in that example completely polarized by* S. Some restriction must be imposed on the decomposition $\underline{g} \oplus \underline{h}$; the one that does the job is

[T] For a (necessarily open) set $P \subset \underline{h}^{\perp}$, $\alpha \in P$ implies

$$\underline{g}_\alpha := \{x \in \underline{g} : \mathrm{ad}'x(\alpha) = 0\} = \{0\}$$

$$\underline{h}_\alpha := \{z \in \underline{h} : \mathrm{ad}'z(\alpha) = 0\} = \{0\} \quad .$$

Remarks.

1) The ad' appearing in [T] refers to $\underline{\ell}$, so $\underline{g}_\alpha = \underline{g} \cap \underline{\ell}_\alpha$, $\underline{h}_\alpha = \underline{h} \cap \underline{\ell}_\alpha$. So the hypothesis may be rephrased: $\underline{\ell}_\alpha$ intersects both $\underline{g}$ and $\underline{h}$ transversally. (hence "T").

2) Situations midway between example 7 and hypothesis [T] have not been explored at all. All the examples of §3 except #7 satisfy [T].

The first point to make is that [T] means roughly that functions in S have as many independent differentials as possible when restricted to $O^G_\alpha \cap R \cap P$, in a local sense. In fact, if $\alpha \in R$, then the Ad'_L-orbit through α is cut out by the locally $\mathrm{ad}'_{\underline{\ell}}$-invariant functions $S(U)$, which have $\dim \underline{\ell}_\alpha$ independent differentials, according to Lemma 4.1. Hence the codimension of O^L_α in $\underline{\ell}^*$ is $\dim \underline{\ell}_\alpha$. According to the next lemma, if $\alpha \in R \cap P$, then the codimension of $O^L_\alpha \cap O^G_\alpha$ in O^G_α is still $\dim \underline{\ell}_\alpha$:

Lemma 4.2

Suppose [T] holds, and let $\alpha \in R \cap P$. Then

$$\dim T_\alpha O^G_\alpha = \dim T_\alpha O^G_\alpha \cap T_\alpha O^L_\alpha + \dim \underline{\ell}_\alpha \quad .$$

Proof: Denote

$$T_\alpha O^L_\alpha = W_\alpha = \{\mathrm{ad}'w(\alpha) : w \in \underline{\ell}\}$$

$$T_\alpha O^G_\alpha = V_\alpha = \{\mathrm{ad}'_{\underline{g}}x(\alpha) = \Pi_{\underline{h}^{\perp}}\mathrm{ad}'x(\alpha) : x \in \underline{g}\} \quad .$$

Using Lemma 4.1 for a suitable open $U \ni \alpha$,

$$V_\alpha \cap W_\alpha = \{\beta \in V_\alpha : dF(\alpha,\beta) = 0 \ \forall F \in \mathcal{S}(U)\}$$

so

$$\dim V_\alpha \cap W_\alpha = \dim V_\alpha - \dim\{dF\big|_{T_\alpha \mathcal{O}_\alpha^G} : F \in \mathcal{S}(U)\} \ .$$

Since $\mathcal{O}_\alpha^G$ is a symplectic manifold,

$$\dim \{dF\big|_{T\mathcal{O}_\alpha^G} : F \in \mathcal{S}(U)\}$$

$$= \dim \{\mathcal{D}_F(\alpha) : F \in \mathcal{S}(U)\}$$

$$= \dim \{-\mathrm{ad}'(\Pi_{\underline{h}} \nabla F(\alpha))(\alpha) : F \in \mathcal{S}(U)\}$$

according to Cor. 2.3. According to Lemma 4.1, this is the same as

$$= \dim \{\mathrm{ad}'\Pi_{\underline{h}} x(\alpha) : x \in \underline{\ell}_\alpha\} \ .$$

Suppose $\mathrm{ad}'\Pi_{\underline{h}} x(\alpha) = 0$ for $x \in \underline{\ell}_\alpha$. Then $\Pi_{\underline{h}} x \in \underline{h}_\alpha$. But by [T], (ii), $\Pi_{\underline{h}} x = 0$ in that case, so $x \in \underline{g} \cap \underline{\ell}_\alpha$. Then [T], (i) implies $x = 0$. So the linear map $x \mapsto \mathrm{ad}'\Pi_{\underline{h}} x(\alpha)$ is injective, hence the above is

$$= \dim \underline{\ell}_\alpha \ .$$

q.e.d.

__Remark__. By Lemma 4.1, for a suitable open neighborhood U of $\alpha \in R \cap P$, the set $\mathcal{O}_\alpha^G \cap \mathcal{O}_\alpha^L \cap U$ is cut out by $\mathcal{S}(U)\big|_{\mathcal{O}_\alpha^G}$, and is a manifold (since the number of independent differentials remains constant over U, and equal to $\dim \underline{\ell}_\alpha$). On the other hand, by Cor. 2.3, this manifold is coisotropic in $\mathcal{O}_\alpha^G$, being cut out by a Poisson - commuting family of functions. Hence its dimension is $\geq \dim \underline{\ell}_\alpha$, and $\mathcal{O}_\alpha^G \cap U$ is completely polarized by $\mathcal{S}(U)$ if and only if

$$\dim \mathcal{O}_\alpha^G \cap \mathcal{O}_\alpha^L \cap U = \dim \underline{\ell}_\alpha$$

which is equivalent to

$$\dim W_\alpha \cap V_\alpha = \dim \underline{\ell}_\alpha \ .$$

Call an orbit of G in $\underline{h}^\perp$ __co-regular__ if it contains a point in $R \cap P$. Then the above argument shows that the co-regular orbits completely polarized by $\mathcal{S}(U)$ in some neighborhood U

of $\alpha \in R \cap P$ are of minimal dimension amongst all co-regular orbits.

The actions of $\underline{g}$ and $\underline{h}$, as subalgebras of $\underline{\ell}$, on $\underline{\ell}^*$ produce another pair of subspaces

$$U_\alpha^{\underline{g}} = \{ad'x(\alpha) : x \in \underline{g}\}$$

$$U_\alpha^{\underline{h}} = \{ad'z(\alpha) : z \in \underline{h}\} .$$

Note that $W_\alpha = U_\alpha^{\underline{g}} + U_\alpha^{\underline{h}}$ (not direct), and $U_\alpha^{\underline{h}} \subset \underline{h}^\perp$ if $\alpha \in \underline{h}^\perp$.

The main result of this section is

Theorem 4.3.

Suppose [T] holds, and $\alpha \in R \cap P$. Then $S(U)$ defines a complete polarization of $\mathcal{O}_\alpha^G \cap U$ for some neighborhood U of α if and only if

$$U_\alpha^{\underline{g}} \cap V_\alpha = U_\alpha^{\underline{h}} \cap V_\alpha .$$

Proof. Thanks to an elementary result from symplectic geometry, $S(U)$ will completely polarize $\mathcal{O}_\alpha^G$ if and only if the tangent spaces to the common level surfaces are spanned by the Hamiltonian vector fields of functions in $S(U)$. The common level surfaces are exactly the sets $\mathcal{O}_\alpha^G \cap \mathcal{O}_\alpha^L$, according to Lemma 4.1. At $\alpha \in R \cap P$ this means

$$\begin{aligned} W_\alpha \cap V_\alpha &= \{-ad'(\Pi_{\underline{h}} \nabla F(\alpha))(\alpha) : F \in S(U)\} \\ &= \{ad'(\Pi_{\underline{g}} \nabla F(\alpha))(\alpha) : F \in S(U)\} \\ &= \{ad'\Pi_{\underline{h}} w : w \in \underline{\ell}_\alpha\} \\ &= \{ad'\Pi_{\underline{g}} w : w \in \underline{\ell}_\alpha\} \end{aligned}$$

by Cor. 2.3 and Lemma 4.1.

Now suppose that $S(U)$ completely polarizes $\mathcal{O}_\alpha^G$ near α, i.e. the above formulae hold, and suppose $z \in \underline{h}$ so that

$$ad'z(\alpha) \in U_\alpha^{\underline{h}} \cap V_\alpha .$$

Certainly $U_\alpha^{\underline{h}} \cap V_\alpha \subset W_\alpha \cap V_\alpha$, so

$$ad'z(\alpha) = ad'\Pi_{\underline{h}}w(\alpha) = -ad'\Pi_{\underline{g}}w(\alpha) \in U_\alpha^{\underline{g}} \cap V_\alpha$$

for some $w \in \underline{\ell}_\alpha$. Thus $U_\alpha^{\underline{h}} \cap V_\alpha \subset U_\alpha^{\underline{g}} \cap V_\alpha$. The opposite inclusion is established the same way, so

$$U_\alpha^{\underline{g}} \cap V_\alpha = U_\alpha^{\underline{h}} \cap V_\alpha \quad .$$

Suppose conversely that this last equality holds, and select $w \in \underline{\ell}$ with $ad'w(\alpha) \in V_\alpha$, i.e. $ad'w(\alpha) \in W_\alpha \cap V_\alpha$. Set $w = x+z$, $x \in \underline{g}$, $z \in \underline{h}$. Then $ad'x(\alpha) = ad'w(\alpha) - ad'z(\alpha) \in \underline{h}^\perp$ since $ad'w(\alpha) \in V_\alpha \subset \underline{h}^\perp$ and $ad'z(\alpha) \in U_\alpha^{\underline{h}} \subset \underline{h}^\perp$, α being in $\underline{h}^\perp$. However this means that

$$ad'x(\alpha) = \Pi_{\underline{h}^\perp} ad'x(\alpha) = ad'_{\underline{g}}x(\alpha) \in V_\alpha \quad .$$

Hence $ad'x(\alpha) \in U_\alpha^{\underline{g}} \cap V_\alpha$. The assumption means that there must be some $z_1 \in \underline{h}$ with $ad'z_1(\alpha) \in U_\alpha^{\underline{h}} \cap V_\alpha$ and $ad'x(\alpha) = ad'z_1(\alpha)$. But then $ad'w(\alpha) = ad'(z+z_1)(\alpha) \in U_\alpha^{\underline{h}} \cap V_\alpha$. Using the assumption again, there is also $x_1 \in \underline{g}$ such that $ad'w(\alpha) = ad'x_1(\alpha)$. Then of course $z + z_1 - x_1 \in \underline{\ell}_\alpha$, so you have proved that

$$W_\alpha \cap V_\alpha \subset \{ad'\Pi_{\underline{h}}u(\alpha) : u \in \underline{\ell}_\alpha\} \quad .$$

But the other inclusion is trivial, and, using [T] as in the proof of Lemma 4.1, you show

$$\begin{aligned} \dim W_\alpha \cap V_\alpha &= \dim \{ad'\Pi_{\underline{h}}u(\alpha) : u \in \underline{\ell}_\alpha\} \\ &= \dim \Pi_{\underline{h}}\underline{\ell}_\alpha \\ &= \dim \underline{\ell}_\alpha \quad . \end{aligned}$$

q.e.d.

Call α a $G - L$ point if $\alpha \in R \cap P$ and

$$U_\alpha^{\underline{g}} \cap V_\alpha = U_\alpha^{\underline{h}} \cap V_\alpha \quad .$$

Cor. 4.4. Suppose α is a $G - L$ point. Then $\mathcal{O}_\alpha^G \cap R \cap P$ consists of $G - L$ points.

Proof: Count dimensions.

q.e.d.

Theorem 4.5.

Suppose α is a G - L point. Then there exist a neighborhood $\mathcal{U}$ of α and real analytic maps

$$X_\alpha : \mathcal{U} \cap \mathcal{O}^L_\alpha \cap \mathcal{O}^G_\alpha \to G$$

$$Y_\alpha : \mathcal{U} \cap \mathcal{O}^L_\alpha \cap \mathcal{O}^G_\alpha \to H$$

so that for any $\beta \in \mathcal{U} \cap \mathcal{O}^L_\alpha \cap \mathcal{O}^G_\alpha$,

$$\beta = \mathrm{Ad}'X_\alpha(\beta)(\alpha) = \mathrm{Ad}'Y_\alpha(\beta)(\alpha)$$

where both Ad' - actions are of L on $\underline{\ell}^*$.
The proof depends on the following:

Lemma 4.6.

Let $\underline{g}$ be a real Lie algebra, G the corresponding connected group. Then:

(i) Suppose $x : [0,1] \to \underline{g}$ is a C^1 map. Then the initial value problem

$$\dot{X}(t) = \xi(t, X(t)) \quad , \quad X(0) = I$$

with

$$\xi(t,X) := dL_X(x(t))$$

has global solutions (i.e. defined on all of $[0,1]$) of class $C^2([0,1];G)$.

(ii) Suppose additionally that x depends continuously (or C^k, or analytically) on some parameters. Then the solution to the above initial value problem depends continously (or C^k, or analytically) on the parameters, as a map into $C^2([0,1];G)$.

Proof (of Lemma). Using standard theorems on existence uniqueness, and continuous dependence on parameters for initial value problems, and a compactness argument, you show that there is $\epsilon > 0$ so that the initial value problem

$$\dot{X}(t) = dL_{X(t)}(x(t+s)) \quad , \quad X(0) = I$$

has a C^2 solution on $-\epsilon < t < \epsilon$ for all $s \in [0,1]$. Call this solution X_s.

Let $Y \in G$, $t_0 \in [0,1]$ and set

$$Z(t) = YX_{t_0}(t-t_0) \quad , t_0 - \epsilon < t < t_0 + \epsilon \quad .$$

The $Z(t_0) = Y$, $Z \in C^2$, and

$$\begin{aligned}
\frac{d}{dt} Z(t) &= dL_Y \frac{d}{dt} X_{t_0}(t-t_0) \\
&= dL_Y \, dL_{X_{t_0}(t-t_0)} (x(t)) \\
&= dL_{YX_{t_0}(t-t_0)} (x(t)) \\
&= dL_{Z(t)} (x(t)) \quad .
\end{aligned}$$

This shows that any solution of the problem

$$X(t) = \xi(t, X(t)) \quad , X(0) = I$$

defined on $[0,\delta]$, $\delta \leq 1$, may be extended to $[0, \min(\delta + \epsilon/2, 1)]$; hence a global solution exists.

(ii) follows from standard theorems on dependence on parameters.

q.e.d.

<u>Proof of Theorem 4.5:</u>

According to <u>Cor. 4.4</u>, $\mathcal{O}_\alpha^G \cap R \cap P$ consists entirely of $G - L$ points, so $\dim(T_\beta \mathcal{O}_\alpha^G \cap T_\beta \mathcal{O}_\beta^L) = \dim \underline{\mathfrak{C}}_\alpha$ for $\beta \in \mathcal{O}_\alpha^G \cap R \cap P$. In particular, this intersection has constant rank, so $\mathcal{O}_\alpha^G \cap \mathcal{O}_\alpha^L \cap R \cap P$ is a submanifold of $\mathcal{O}_\alpha^G \cap R \cap P$. Since $\mathcal{O}_\alpha^G$ and $\mathcal{O}_\alpha^L$ are real-analytic (see [20], Ch. 2) so is the intersection. Also there is some neighborhood $U \subset R \cap P$ of α so that $U \cap \mathcal{O}_\alpha^G \cap \mathcal{O}_\alpha^L$ is arc-wise connected and simply connected.

Select $\beta \in U \cap \mathcal{O}_\alpha^G \cap \mathcal{O}_\alpha^L$, and denote by $\gamma : [0,1] \to U \cap \mathcal{O}_\alpha^G \cap \mathcal{O}_\alpha^L$ any C^2 - arc with $\gamma(0) = \alpha$, $\gamma(1) = \beta$. Since $\dot{\gamma}(t) \in T_{\gamma(t)}(\mathcal{O}_\alpha^G \cap \mathcal{O}_\alpha^L)$, there must exist $x(t) \in \underline{g}$, $y(t) \in \underline{h}$ for which (Theorem 4.3)

$$\dot{\gamma}(t) = ad'x(t) \ (\gamma(t)) = ad'y(t) \ (\gamma(t))$$

according to Lemma 4.1. We claim that $x : [0,1] \to \underline{g}$, $y : [0,1] \to \underline{h}$ are C^1 maps. In fact, because of [T] the map

$$x \to ad'x(\beta)$$

is a linear isomorphism of $\underline{g}$ onto a subspace of $T_\beta \mathcal{O}^L_\beta$, for $\beta \in R \cap P$. Also, the collection of such subspaces

$$E_{\underline{g}} = \{(\beta,\ ad'x(\beta)) : x \in \underline{g},\ \beta \in R \cap P\}$$

is a smooth subbundle of $T\underline{\ell}^*\big|_{R\cap P}$. Hence the map $A_{\underline{g}} : E_{\underline{g}} \to \underline{g}$ defined by

$$A_{\underline{g}}((\beta,\ ad'x(\beta)) = x$$

is well-defined and smooth. Since

$$x(t) = A_{\underline{g}}(\dot{\gamma}(t))$$

and $\dot{\gamma}$ is a C^1 map, it follows that x is C^1; similarly y is C^1.

Set $\xi(t,X) = dL_X(x(t))$.

According to Lemma 4.6, the initial value problem

$$\dot{X}(t) = \xi(t,\ X(t)) \quad , \quad X(0) = I$$

has a solution in $C^2([0,1];G)$. Set

$$\tilde{\gamma}(t) = Ad'(X(t))(\alpha) \quad .$$

Then $\tilde{\gamma}(0) = \alpha$, and

$$\begin{aligned} \dot{\tilde{\gamma}}(t) &= ad'(dL_{X^{-1}(t)}(\dot{X}(t)))(\tilde{\gamma}(t)) \\ &= ad'(dL_{X^{-1}(t)}\ \xi(t,\ X(t))(\tilde{\gamma}(t)) \\ &= [ad'(x(t))]\ (\tilde{\gamma}(t)) \quad . \end{aligned}$$

Since γ and $\tilde{\gamma}$ obey the same differential equation and have the same initial point, they coincide. In particular

$$\gamma(1) = \beta = Ad'X(1)(\alpha)$$

so set $X_\alpha(\beta) \equiv X(1)$.

Similarly, $Y_\alpha(\beta) \equiv Y(1)$, where Y is the solution of

$$\dot{Y}(t) = \eta(t,Y(t))$$

$$Y(0) = 1$$

$$\eta(t,Y) = DL_Y(y(t)) \quad .$$

These definitions make sense only if the choice of path γ is immaterial. To see that this is the case, let $\bar{\gamma} : [0.1] \to \mathcal{U} \cap \mathcal{O}^L_\alpha \cap \mathcal{O}^G_\alpha$ be a C^2 path with $\bar{\gamma}(0) = \alpha$ and $\bar{\gamma}(1) = \beta$. Reasoning as before, obtain an expression

$$\beta = \mathrm{Ad}'\bar{X}(1)(\alpha)$$

for some path $\bar{X}$ in G. In particular,

$$X(1)^{-1}\bar{X}(1) \in G^L_\alpha \quad ,$$

where

$$G^L_\alpha = \{X \in G : \mathrm{Ad}'X(\alpha) = \alpha\}$$

is the isotropy subgroup of $\alpha \in \underline{g}^* = h^\perp$ under the Ad' action of L on $\underline{\ell}^*$, restricted to G: that is, $G^L_\alpha = L_\alpha \cap G$. We will show that $X(1)^{-1}\bar{X}(1)$ is in the connected component of I in G^L_α; since the Lie algebra of G^L_α is $\underline{g}_\alpha = \{0\}$ according to the transversality hypothesis, the connected component of I is $\{I\}$, hence $X(1) = \bar{X}(1)$ as desired.

Since $\mathcal{U} \cap \mathcal{O}^L_\alpha \cap \mathcal{O}^G_\alpha$ is simply connected, there is a C^2 homotopy

$$\Gamma : [0,1] \times [0,1] \to \mathcal{U} \cap \mathcal{O}^L_\alpha \cap \mathcal{O}^G_\alpha$$

with

$$\Gamma(t,0) = \gamma(t) \qquad \Gamma(1,s) = \beta$$

$$\Gamma(t,1) = \bar{\gamma}(t) \qquad \Gamma(0,s) = \alpha$$

$$0 \le t,\ s \le 1 \quad .$$

You apply the construction explained above to each path $t \mapsto \Gamma(t,s)$ to obtain a C^2 homotopy

$$\chi : [0,1] \times [0,1] \to G$$

with $\chi(t,0) = X(t)$, $\chi(t,1) = \bar{X}(t)$. In particular

$$\beta = \mathrm{Ad}'\chi(1,s)(\alpha) \quad , \quad 0 \le s \le 1$$

so that

$$\chi(1,s_1)^{-1}\chi(1,s_2) \in G_\alpha^L$$

for $s_1, s_2 \in [0,1]$. However, the image of the connected set $[0,1] \times [0,1]$ under the continuous map

$$(s_1,s_2) \mapsto \chi(1,s)^{-1}\chi(1,s_2)$$

is connected, hence lies in a component of G_α^L. Since $\chi(1,0)^{-1}\chi(1,0) = I$, you conclude that this must be the component of I, namely $\{I\}$. In particular, $\chi(1,0)^{-1}\chi(1,1) = X^{-1}(1)\bar{X}(1) = I$, as was asserted.

This shows that the map X_α is well-defined in $\mathcal{U} \cap \mathcal{O}_\alpha^L \cap \mathcal{O}_\alpha^G$. Similarly the map Y_α is well-defined.

The analyticity of X_α and Y_α follows from (ii) of Lemma 4.6.

q.e.d.

Since $\underline{\ell} = \underline{g} \oplus \underline{h}$, there is some neighborhood $\mathcal{U}$ of $I \in L$ and unique analytic maps $\mathcal{X} : \mathcal{U} \to G$ and $\mathcal{Y} : \mathcal{U} \to H$ so that for any $X \in \mathcal{U}$, $X = \mathcal{X}(X)\mathcal{Y}^{-1}(X)$. In particular, $\mathcal{X}(I) = \mathcal{Y}(I) = I$.

Cor. 4.7. The following diagram commutes

$$\begin{tikzcd}
& L_\alpha \cap \mathcal{U} \arrow[dl, "\mathcal{X}"'] \arrow[dr, "\mathcal{Y}"] \arrow[dd, "\mathrm{Ad}'\mathcal{Y}(\cdot)(\alpha)"'] & \\
G & & H \\
& \mathcal{O}_\alpha^L \cap \mathcal{O}_\alpha^G \arrow[ul, "X_\alpha"] \arrow[ur, "Y_\alpha"'] &
\end{tikzcd}$$

§5. Norming Constants

This section explains another set of functions on $\underline{h}^{\perp}$, complementary to the Ad-invariant class functions, when $\underline{\ell}$ is a normal real form of a complex semisimple Lie algebra, decomposed according to the compact-Borel decomposition as in §3, Example 1. These are a direct analog of the "norming constants" of Sturm-Liouville theory (or rather, their reciprocal squares). They are equal in number to the rank of $\underline{\ell}$, and are denoted β below. The immediate ancestors of the β's are the residues r_i^2 of Moser [2], which in turn date back to the work of Stieltjes. The main fact about the norming constants is that they form, together with certain convenient Ad-invariant functions generalizing the eigenvalues of symmetric matrices, a system of functions with $2r$ independent differentials at "most" points of an orbit of G in $\underline{h}^{\perp}$; this is Prop 5.11. This result gives a complete set of coordinates on a "completely polarized" orbit (Cor. 5.12). Curiously, we have not been able to prove these results without resort to representation theory, even though the definition of the β's has nothing to do with representations of $\underline{\ell}$. On the other hand, the irreducible representations of $\underline{\ell}$ will be needed to solve the various completely integrable systems, implied in this situation by Theorem 2.2, by means of rational functions of exponentials (§6). An appendix to this section exposes the basic facts about the "highest weight" construction of irreducible representations of $\underline{\ell}$ via the Verma Module, and uses this construction in a proof of the existence of a unique Euclidean structure on each irreducible representation for which the Cartan involution σ is represented by the metric adjoint. This can be done more concisely by integration over the compact subgroup H, but perhaps the algebraic proof given here (which seems to be new) might be useful in contexts where the analytic machinery is not available.

Let $\underline{\ell}$ be a normal real form of a complex semisimple Lie algebra as in §3, Example 1, and let $\underline{\ell} = \underline{g} \oplus \underline{h}$ be the compact/Borel decomposition relative to a vector Cartan Subalgebra $\underline{c}$. According to fact (6) listed in §3.1, $\underline{h}^{\perp}$ consists of semisimple elements, so for $a \in \underline{h}^{\perp}$,

$$\underline{\ell} = \underline{\ell}_a \oplus \text{Range (ad } a) \quad .$$

(Remember that $\underline{\ell}$ and $\underline{\ell}^*$ are identified by the Killing form K. In honor of this identification elements of $\underline{h}^{\perp}$ will henceforth be denoted by small Latin, rather than Greek, letters.

In the same spirit, R will henceforth denote the set of regular points of $\underline{\ell}$. Occasionally, an element of $\underline{\ell}$ will be denoted also by a Greek letter when viewed as an element of $\underline{\ell}^*$, viz. c and ζ below. The reader will be warned by the phrase "dual to ... under K".)

<u>Lemma 5.1</u>

(i) Suppose $\underline{j} \subset \underline{\ell}$ is a CSA and $a \in \underline{j} \cap \underline{h}^\perp$ is regular. Then $\underline{j} \subset \underline{h}^\perp$.

(ii) Suppose $a \in \underline{h}^\perp$ is regular. Then there exists a neighborhood U of the CSA $\underline{\ell}_a$ in $\underline{h}^\perp$ so that for any regular $b \in U$, $\underline{\ell}_b = \mathrm{Ad}\, Y(\underline{\ell}_a)$ for some $Y \in H$.

<u>Proof</u>: (i) Under the hypotheses, $\underline{j} = \underline{\ell}_a$. Clearly

$$\dim \ker (\mathrm{ad}\, a\big|_{\underline{h}^\perp}) \le r = \mathrm{rank}\, \underline{c} \quad .$$

Now $\dim \underline{\ell} - r$ is even, and

$$\dim \underline{h}^\perp = \frac{1}{2}(\dim \underline{\ell} + r) \quad \dim \underline{h} = \frac{1}{2}(\dim \underline{\ell} - r) \quad .$$

Since $[\underline{h}^\perp, \underline{h}^\perp] \quad \underline{h}$,

$$\dim \ker (\mathrm{ad}\, a\big|_{\underline{h}^\perp}) + \dim \mathrm{Range}\, (\mathrm{ad}\, a\big|_{\underline{h}^\perp}) = \dim \underline{h}^\perp$$

$$\le r + \dim \underline{h}$$

and equality can only be attained - as it must be - if

$$\dim \ker(\mathrm{ad}\, a\big|_{\underline{h}^\perp}) = r$$

$$\dim \mathrm{Range}(\mathrm{ad}\, a\big|_{\underline{h}^\perp}) = \dim \underline{h} \quad .$$

The first equality implies $\underline{j} = \underline{\ell}_a \subset \underline{h}^\perp$, and the second implies that $\underline{h} = \mathrm{Range}(\mathrm{ad}\, a\big|_{\underline{h}^\perp})$.

(ii) Since $[\underline{h}^\perp, \underline{h}] \subset \underline{h}^\perp$, $[\underline{h}^\perp, \underline{h}^\perp] \subset \underline{h}$, it is clear that

$$\mathrm{Range}(\mathrm{ad}\, a) \cap \underline{h}^\perp = \mathrm{Range}(\mathrm{ad}\, a\big|_{\underline{h}})$$

so $\underline{h}^\perp = \underline{\ell}_a \oplus \mathrm{Range}(\mathrm{ad}\, a\big|_{\underline{h}})$ for $a \in \underline{h}^\perp \cap R$. This implies, by elementary arguments, that

there is some neighborhood V of the identity in H, and a neighborhood U of a, so that U consists of regular points b for which $b = \mathrm{Ad}\, Y(a')$, some $Y \in V$, $a' \in \underline{\ell}_a$. But then

$$\bigcup_{b \in U} \underline{\ell}_b = \;=\; \bigcup_{Y \in V} \mathrm{Ad}\, Y(\underline{\ell}_a)$$

is a neighborhood of $\underline{\ell}_a$.

q.e.d.

Cor. 5.2. The compact/Borel decomposition $\underline{\ell} = \underline{g} \oplus \underline{h}$ satisfies [T].

Proof: It's clear from Lemma 5.1 that $\underline{\ell}_a \cap \underline{h} = \{0\}$ for regular $a \in \underline{h}^{\perp}$. On the other hand, $\underline{g} \cap \underline{h}^{\perp} = \underline{c}$. Since a regular element a is contained in a unique CSA $\underline{\ell}_a = \underline{c}(a) \subset h^{\perp}$, $\underline{g} \cap \underline{\ell}_a = \underline{c} \cap \underline{c}(a) = \{0\}$ unless $a \in \underline{c}$. Therefore you may take for P in the notation of the statement of [T] in §4 the set

$$P = (\underline{h}^{\perp} \setminus \underline{c}) \cap R$$

which is open and dense as required.

q.e.d.

Remark. As noted in the introduction, an orbit is coregular in this context if it contains a regular point $a \notin \underline{c}$.

Lemma 5.3

(i) Let $a \in \underline{h}^{\perp}$. Then a is contained in a CSA of $\underline{\ell}$ contained in $\underline{h}^{\perp}$.

(ii) $\underline{h}^{\perp}$ is the Ad H-orbit of $\underline{c}$.

Proof: Select $a \in \underline{h}^{\perp}$; then a is in the boundary of an arc-wise connected subset of $\underline{h}^{\perp} \cap R$. Let $a_n \in \underline{h}^{\perp} \cap R$ so that $a_n \to a$. By connecting successive a_n's with arcs in $\underline{h}^{\perp} \cap R$ and using Lemma 5.1 and an easy compactness argument, you deduce the existence of a sequence $\{Y_n\}$ H so that $\underline{\ell}_{a_n} = \mathrm{Ad}\, Y_n(\underline{\ell}_{a_1})$.

Set $b_n = \mathrm{Ad}\, Y_n^{-1}(a_n) \in \underline{\ell}_{a_1}$. Since H is compact, $\{Y_n^{-1}\}$ has a convergent subsequence. Denote it also by $\{Y_n^{-1}\}$. Since Ad is continuous, $\mathrm{Ad}\, Y_n^{n}(a_n) = b_n \to b \in \underline{\ell}_{a_1}$, $\underline{\ell}_{a_1}$ being closed. Let $Y = \lim_{n \to \infty} Y_n$ and select a neighborhood V of 0 in $\underline{\ell}$.

The convergence of the adjoint representatives of a convergent subsequence is uniform on compacta, so there is a neighborhood U_V of b in $\underline{h}^1$ and an integer N_V so that

$$\mathrm{Ad}\, Y_n(x) - \mathrm{Ad}\, Y(x) \in V \quad \text{for}$$

$x \in U_V$, $n > N_V$.

On the other hand there is another integer M_V so that $b_n \in U_V$ for $n > M_V$, hence

$$\mathrm{Ad}\, Y_n(b_n) - \mathrm{Ad}\, Y(b_n) \in V \quad \text{for}$$

$n > \max(N_V, M_V)$. Thus (V being arbitrary)

$$\mathrm{Ad}\, Y_n(b_n) - \mathrm{Ad}\, Y(b_n) \to 0 \qquad n \to \infty \quad .$$

On the other hand

$$\mathrm{Ad}\, Y(b_n) - \mathrm{Ad}\, Y(b) \to 0 \qquad n \to \infty$$

so

$$\mathrm{Ad}\, Y_n(b_n) - \mathrm{Ad}\, Y(b) \to 0 \qquad n \to \infty \quad .$$

However, $\mathrm{Ad}\, Y_n(b_n) = a_n \to a$, so $\mathrm{Ad}\, Y(b) = a$. Therefore a is contained in the CSA $\mathrm{Ad}\, Y(\underline{\ell}_{a_1}) \subset \underline{h}^1$, and (i) is proved.

Since $\underline{h}^1$ is the union of the CSA's contained in it, the quotient space obtained by identifying elements of $\underline{h}^1$ belonging to the same CSA is connected and locally compact.

Lemma 5.1 (ii) may be interpreted as asserting that Ad H acts locally transitively on this quotient space. Then a simple compactness argument establishes (ii).

q.e.d.

Remark. By a very similar argument, you can show that each $a \in \underline{h}^1$ is a member of a unique CSA $\underline{c}(a) \subset \underline{h}^1$.

Lemma 5.4

For any CSA $\underline{j} \subset \underline{h}^1$ the isotropy subgroup $H_j = \{X \in H : \mathrm{Ad}\, X(j) = j\}$ is discrete, (in fact, finite).

Proof: First note that $H_{\underline{j}}$ is a closed subgroup of H - which is obvious, since $\underline{j}$ is a closed set. By a basic theorem of Lie Theory, $H_{\underline{j}}$ is a Lie subgroup, so it suffices to observe that it has a trivial Lie algebra. But this is clear, since its Lie algebra is contained in the normalizer of $\underline{j}$, which is $\underline{j}$, and $\underline{j}$ and $\underline{h}$ intersect transversally.

q.e.d.

If $\underline{j} \subset \underline{h}^{\perp}$ is a CSA, there is some neighborhood $\tilde{U}(\underline{j})$ of $\underline{j}$ in the space of CSA's in $\underline{h}^{\perp}$ and a neighborhood $V(\underline{j})$ of I in H so that $\underline{j}' \in \tilde{U}(\underline{j})$ has a unique expression (Lemma 5.4) as

$$\underline{j}' = \operatorname{Ad} Y(\underline{j},\underline{j}')(\underline{j}) \quad ,$$

$$\text{with } Y(\underline{j},\underline{j}') \in V(\underline{j}) \quad .$$

Select a basis $\phi_1, \ldots \phi_r$ of $\underline{j}$. For $a \in \underline{j}'$, $\underline{j}' \in \tilde{U}(\underline{j})$, set

$$\psi_i(a) = \operatorname{Ad} Y(\underline{j},\underline{j}')(\phi_i), \quad i = 1,\ldots,r \quad ,$$

$\{\psi_i : U(\underline{j}) \to \underline{h}^{\perp}\ i = 1,\ldots,r\}$ is an Ad-invariant choice of basis for the CSA's in $U(\underline{j})$, where $U(\underline{j})$ is the neighborhood in $\underline{h}^{\perp}$ covering $\tilde{U}(\underline{j})$. Fix $\underline{j}$ for the moment, and write $U(\underline{j}) = U$, $V(\underline{j}) = V$.

For an arbitrary $c \in \underline{\ell}$ and regular $a \in U$ you can write

$$c = \beta_1(a)\psi_1(a) + \cdots + \beta_r(a)\psi_r(a)$$
$$+ [\xi(a), a] \quad .$$

The β's are uniquely determined as functions on U by c, and are constant as a varies along a fixed CSA; ξ may be fixed by requiring $\xi(a) \in \operatorname{Range}(\operatorname{ad} a)$. The next three Lemmas are devoted to the computation of $\nabla\beta_i$.

Lemma 5.5. The functions

$$\lambda_i(a) = K(\psi_i(a), a)$$

are ad-invariant. The polynomial ring in $\{\lambda_i\}_{i=1,\ldots,r}$ contains the ring of restrictions of Ad-invariant polynomials to U.

Proof: By definition, for $a \in U$, $Y \in H$ close to the identity.

$$\psi_i(\mathrm{Ad}\, Y(a)) = \mathrm{Ad}\, Y(\psi_i(a)) = \mathrm{Ad}\, Y(\phi_i)$$

so $K(\psi_i(\mathrm{Ad}\, Y(a)), \mathrm{Ad}\, Y(a)) = K(\psi_i(a), a)$ which shows that $\lambda_1 \ldots \lambda_r$ are Ad-invariant. For the second assertion, note that for each $a \in \underline{h}^{\perp}$, $\{\lambda_i\}$ are affine coordinate functions on $\underline{c}(a)$, so the polynomial ring $\mathbb{R}[\lambda_1 \ldots \lambda_r]$ is the polynomial ring on $\underline{c}(a)$. According to Chevalley's theorem ([23], Thm. 23.1) the restrictions of the Ad-invariant polynomials to $\underline{c}(a)$ are exactly the polynomials on $\underline{c}(a)$ invariant with respect to the Weyl group of the pair $(\underline{\ell}, \underline{c}(a))$. Since the basis $\{\psi_i\}$ is Ad-invariant, it's clear that the coefficients of powers of $\lambda_1, \ldots \lambda_r$ in the restriction of a given Ad-invariant polynomial to $\underline{c}(a)$ are actually independent of a, which finishes the proof.

q.e.d.

Lemma 5.6. $\nabla\lambda_i(a) = \psi_i(a)$, $a \in U$.

Proof: Suppose that a is regular. Select $b \in \underline{c}(a)$, $x \in \underline{h}$. Then

$$d\lambda_i(a, b + [a,x]) = d\lambda_i(a, b)$$
$$= K(d\psi_i(a,b), a) + K(\psi_i(a), b)$$
$$= K(\psi_i(a), b) = K(\psi_i(a), b + [a,x])$$

since the ψ's are constant on $\underline{c}(a)$ and orthogonal to Range ad a.

For general $a \in \underline{h}^{\perp}$, the same formula holds by continuity.

q.e.d.

Denote by Λ the inverse of the (positive-definite) matrix $K(\psi_i, \psi_j)$ (which is clearly independent of $a \in U$).

Lemma 5.7. $\nabla\beta_i(a) = \sum_{j=1}^{r} \Lambda_{ij}[\xi(a), \psi_j(a)]$.

Proof: Since the β's are constant along $\underline{c}(a)$, a regular, you need only compute $d\beta_i(a, [x,a])$.

Now

$$dK(c, \psi_i)\ (a, [x,a])$$

$$= \sum_{j=1}^{r} (\psi_j, \psi_i)\ d\beta_j\ (a, [x,a]) \quad .$$

However $\psi_i(a + \epsilon[x,a]) = \psi_i(a) + \epsilon \text{ ad } x(\psi_i(a)) + O_2(\epsilon)$, so

$$dK(c,\psi_i)\ (a, [x,a])$$
$$= K(c, [x,\psi_i(a)])$$
$$= -K(x, [c,\psi_i(a)])$$
$$= -K(x, [[\xi(a), \psi_i(a)], a])$$
$$= K([\xi(a),\psi_i(a)], [x,a]) \quad .$$

Now equate the two expressions for $dK(c,\psi_i)$ to obtain the asserted formula.

q.e.d.

Cor. 5.8. The $\{\beta_i\}$ are in involution on $\underline{h}^{\perp}$ in the Poisson structure on $\underline{\ell}^* \cong \underline{\ell}$, that is,

$$\langle a, [\nabla\beta_i(a), \nabla\beta_j(a)]\rangle \equiv 0 \quad .$$

Proof:

$$K(a\ [\nabla\beta_i(a),\ \nabla\beta_j(a)])$$
$$= K(a, [[\xi(a),\psi_i(a)], [\xi(a),\psi_j(a)]])$$
$$= -K([\xi(a),\psi_j(a)], [[\xi(a),\psi_i(a)], a])$$
$$= -K([\xi(a),\psi_j(a)]\ [c,\ \psi_i(a)])$$
$$= K(c, [[\xi(a),\psi_j(a)]\psi_i(a)]) \quad .$$

Since $[\psi_i,\psi_j] = 0$, $[[\xi\ \psi_j]\psi_i] = [[\xi\ \psi_i]\psi_j]$. That is, the last line is symmetric in i,j. Since the first line is clearly antisymmetric, they both vanish.

q.e.d.

This certainly need not mean that the β's are in involution in the $\underline{g}$-coadjoint Poisson structure on $\underline{g}^* \simeq \underline{h}^{\perp}$.

Remark. For $\underline{\ell} = s\ell(n,\mathbb{R})$ you can compute certain quantities $\{r_i^2\}$ related to the β's by linear constant coefficient relations. The $\{r_j\}$, considered by Moser [2] and earlier Stieltjes, satisfy

$$\{r_i^2, r_j^2\} = r_i^2 r_j^2 \left(\frac{r_i^2 + r_j^2}{\lambda_i - \lambda_j}\right)$$

where the λ's are as above for a certain choice $\{\phi_i\}$ of basis in $\underline{c} = d(n,\mathbb{R})$, in fact, the λ's are the eigenvalues of the symmetric matrix $a \in \underline{h}^{\perp}$, where $s\ell(n,\mathbb{R})$ is identified with its fundamental representation. The next point to consider is the relation of the constructions just developed to the various irreducible representations of $\underline{\ell}$. In any case, though, the above formula shows that the β's do not generally form a system in involution.

For the remainder of this section, assume $c \in \underline{c}$. Let $\zeta \in \underline{c}^*$ be dual to c by the Killing form, and denote by $V(\zeta)$ the irreducible representation of $\underline{c}$ with highest weight ζ. Thus

$$V(\zeta) = V(\zeta)_{\zeta} \oplus \sum_{\mu<\zeta} V(\zeta)_{\mu}$$

where $x \cdot v = \mu(x) \cdot v$ for $x \in \underline{c}$ and $v \in V(\zeta)_{\mu}$.

Various facts regarding the highest weight theory are collected for easy reference in the Appendix to this section. For now you need to recall that

1) If ζ is dominant integral - as shall henceforth be required - $\dim V(\zeta) < \infty$.

2) $V(\zeta)_{\zeta} = \mathbb{R} \cdot v_0$, where v_0 is a maximal vector of weight ζ, which shall remain fixed for the remainder of the discussion.

3) There is a Euclidean metric $\langle , \rangle$ on $V(\zeta)$ having the properties:

   (i) $|v_0| = 1$

   (ii) $x^{\dagger} = \sigma(x)$ for $x \in \underline{n}_{\pm}$

   (iii) $\langle V(\zeta)_{\mu}, V(\zeta)_{\nu} \rangle = 0$, $\mu \neq \nu$.

Here $\dagger$ denotes the adjoint of $x \in \operatorname{End} V(\zeta)$ with respect to $\langle,\rangle$. The metric $\langle,\rangle$ is uniquely determined by these requirements. Note that $\underline{h}$ acts by skew-symmetric transformations on $V(\zeta)$, hence its connected group H acts as a subgroup of the orthogonal group of $\langle,\rangle$. Likewise, $\underline{h}^{\perp}$ acts by symmetric transformations on $V(\zeta)$.

4) If $v \in V(\zeta)_{\mu}$, $\tau \in \mathbb{R}$, $x = \tau Z_{\alpha}$ a root vector ($\{Z_{\alpha}\}$ is a Weyl basis of $\underline{c}$, as in §3), then $x \cdot v \in V(\zeta)_{\mu+\alpha}$.

Note that $K(\underline{n}_{\pm},\underline{c}) = 0$. If you denote by $a_{\pm}, a_0$ the components of $a \in \underline{h}^{\perp}$ in the decomposition $\underline{c} = \underline{n}_{-} \oplus \underline{c} \oplus \underline{n}_{+}$, then

$$\begin{aligned} K(c,a) &= K(c, a_0) \\ &= \zeta(a_0) \\ &= \langle v_0, a_0 \cdot v_0\rangle \end{aligned}$$

by (2) and (3)(i) above. By (4), however, $a_{+}v_0 = 0$ and $a_{-}v_0 \in \sum_{\mu<\zeta} V(\zeta)_{\mu}$, so by (3)(iii)

$$= \langle v_0, a \cdot v_0\rangle \quad .$$

Now apply these considerations to the ad-invariant basis $\{\psi_i(a)\}$ in the neighborhood $U(\underline{j})$, together with the notation and proof of Lemma 5.7 above, to obtain

<u>Lemma 5.9</u>. Set $\phi_i(a) = \sum_j \Lambda_{ij}\psi_j(a)$. Then

$$\beta_i(a) = \langle v_0, \phi_i(a) \cdot v_0\rangle \quad .$$

<u>Remark</u>. Note that, apart from the normalization of the maximal vector v_0 implicit in the definition of the metric $\langle,\rangle$, this formula holds independently of the irreducible representation chosen.

For the remainder of this section, ζ will be a fixed dominant integral element of $\underline{c}^*$ and mention of it will be suppressed: thus, write V for $V(\zeta)$, V_{μ} for $V(\zeta)_{\mu}$, and so on. Denote by Φ the (finite) collection of nonzero weights of V, and for $\mu \in \Phi$ let P_{μ} be the orthogonal projection on V_{μ}.

As remarked above, $a \in \underline{h}^{\perp}$ acts on V symmetrically with respect to $\langle , \rangle$ hence is diagonalizable over $\mathbb{R}$ by an orthogonal transformation. In view of Lemma 5.3, $a \in \underline{h}^{\perp}$ is even conjugate to an element of $\underline{c}$ by Ad H. To avoid ambiguities arising from the discrete normalizer of $\underline{c}$ in Ad H, select $a_0 \in \underline{h}^{\perp}$ and $Y_0 \in H$ for which $\mathrm{Ad}\, Y_0(a_0) \in \underline{c}$. As in the set-up preceding the definition of the Ad-invariant basis $\{\psi_i\}$, let U denote a neighborhood of $\underline{c}(a_0)$ in $\underline{h}^{\perp}$, and V a neihborhood of the identity in H, so that for $a \in U$, $a = \mathrm{Ad}\, Y(a,a_0)^{-1}(a')$ for a unique $Y(a,a_0) \in V \subset H$, $a' \in \underline{c}(a_0)$.

For $a \in U$ put

$$P_\mu(a) = Y(a,a_0)^{-1} Y_0^{-1} P_\mu Y_0 Y(a,a_0)$$

and

$$\begin{aligned}\mu^*(a) &= \mu(\mathrm{Ad}[Y_0 Y(a,a_0)](a)) \\ &= K(c_\mu, \mathrm{Ad}[Y_0 Y(a,a_0)](a))\end{aligned}$$

where $c_\mu \in \underline{c}$ is dual to $\mu \in \underline{c}^*$ under K. Then $\{\mu^* : \mu \in \Phi\}$ is another collection of Ad-invariant functions, and $a \cdot P_\mu(a) = \mu^*(a) P_\mu(a)$ so a has the spectral expansion in V:

$$a = \sum_{\mu\in\Phi} \mu^*(a) P_\mu(a) \quad .$$

Any endomorphism of V commuting with a must commute with the $P_\mu(a)$. However, elements of $\underline{c}(a)$ must actually be linear combinations of the $P_\mu(a)$: for $\underline{c}(a) = \underline{c}$, this is merely a restatement of the weight decomposition, and the property is carried to arbitrary $a \in \underline{h}^{\perp}$ by the action of Ad H, thanks to Lemma 5.3. For the ad-invariant basis $\psi_i(a)$ of $\underline{c}(a)$ in U, write

$$\psi_i(a) = \sum_{\mu\in\Phi} \Gamma_i^\mu P_\mu(a) \quad .$$

Also define

$$\begin{aligned}\rho_\mu(a) &\equiv \langle v_0, P_\mu(a) v_0 \rangle \\ &= r_\mu^2(a)\end{aligned}$$

where $r_\mu(a) \equiv |P_\mu(a)v_0|$. Note that $\sum_{\mu\in\Phi} r_\mu^2 = 1$ since $\langle v_0, v_0\rangle = 1$. In view of the above reasoning and definitions, and Lemma 5.9,

Lemma 5.10

(i) $\beta_i(a) = \sum_{j=1}^{r} \sum_{\mu\in\Phi} \Lambda_{ij}\Gamma_j^\mu\rho_\mu(a)$

(ii) $\mu^*(a) = \sum_{j=1}^{r} \tilde{\Gamma}_\mu^j\lambda_j(a)$ where $\{\tilde{\Gamma}_\mu^j\}$ is defined by

$$c_\mu(a) \equiv \mathrm{Ad}(Y^{-1}(a,a_0)Y_0^{-1})(c_\mu)$$

$$= \sum_{j=1}^{r} \tilde{\Gamma}_\mu^j\psi_j(a) \quad .$$

The central result of this section is

Prop. 5.11. The matrix

$$(\{\beta_i,\lambda_j\})_{i,j=1}^{r}$$

is nonsingular on $\underline{h}^\perp \cap R \cap P$, for suitable choice of the highest weight ζ.

Proof. Use Lemmas 5.6, 5.7 to write

$$\{\beta_i,\lambda_j\}(a) = K(a, [\Pi_{\underline{g}}\nabla\beta_i(a), \Pi_{\underline{g}}\nabla\lambda_j(a)])$$

$$= \sum_{k=1}^{r} \Lambda_{ik}K(a, [\Pi_{\underline{g}}[\xi(a),\psi_k(a)], \Pi_{\underline{g}}\psi_j(a)]) \quad .$$

Since Λ is nonsingular, you need only show the non singularity of

$$K(a, [\Pi_{\underline{g}}[\xi(a),\psi_k(a)], \Pi_{\underline{g}}\psi_j(a))$$

$$= -K(a, [\Pi_{\underline{g}}[\xi(a)\psi_k(a)], \Pi_{\underline{h}}\psi_j(a)]) \quad \text{(Lemma 5.5)}$$

$$= K(\Pi_{\underline{g}}[\xi(a),\psi_k(a)], [a, \Pi_{\underline{h}}\psi_j(a)])$$

$$= K([\xi(a),\psi_k(a)], [a, \Pi_{\underline{h}}\psi_j(a)])$$

since $[a, \Pi_{\underline{h}}\psi_j] \in \underline{h}^\perp$

$$= -K(\Pi_{\underline{h}}\psi_j(a), [a, [\xi(a), \psi_k(a)]])$$

$$= -K(\Pi_{\underline{h}}\psi_j(a), [\psi_k(a), c])$$

(recall that $c = \sum \beta_i\psi_i + [\xi, a]$ for $a \in R \cap \underline{h}^{\perp}$)

$$= K(c, [\psi_k(a), \Pi_{\underline{h}}\psi_j(a)])$$

$$= \langle v_0, [\psi_k(a), \Pi_{\underline{h}}\psi_j(a)]v_0\rangle \quad .$$

Since $\underline{h}^{\perp}$ acts symmetrically and h skew-symmetrically on V, this is

$$= 2\langle\psi_k(a)v_0, \Pi_{\underline{h}}\psi_j(a)v_0\rangle \quad .$$

Digress for a moment to consider any $x \in \underline{h}^{\perp}$, written

$$x = \sum_{\alpha\in\Delta^+} x_\alpha(Z_\alpha + Z_{-\alpha}) + x_0$$

where $x_0 \in \underline{c}$, $x_\alpha \in \mathbb{R}$ for $\alpha \in \Delta^+$, and $\{Z_{\pm\alpha}\}$ is the Weyl basis. Then

$$\Pi_{\underline{h}}x = \sum_{\alpha\in\Delta^+} x_\alpha(Z_\alpha - Z_{-\alpha})$$

hence

$$\Pi_{\underline{h}}x \cdot v_0 = -\sum_{\alpha\in\Delta^+} x_\alpha Z_{-\alpha} \cdot v_0$$

since $Z_\alpha \cdot v_0 = 0$, $\alpha \in \Delta^+$. So

$$\begin{aligned}
\Pi_{\underline{h}}x \cdot v_0 &= -x \cdot v_0 + x_0 \cdot v_0 \\
&= -x \cdot v_0 + \zeta(x_0) \cdot v_0 \\
&= -x \cdot v_0 + \langle v_0, x_0 \cdot v_0\rangle v_0 \\
&= -x \cdot v_0 + \langle v_0, x \cdot v_0\rangle \cdot v_0 \quad .
\end{aligned}$$

Apply this result to the last expression preceding the digression to obtain

$$\begin{aligned}
&\langle\psi_k(a)v_0, \Pi_{\underline{h}}\psi_j(a)v_0\rangle \\
&= -\langle\psi_k(a)v_0, \psi_j(a)v_0\rangle + \langle v_0\psi_k(a)v_0\rangle\langle v_0, \psi_j(a)v_0\rangle \quad .
\end{aligned}$$

This symmetric $r \times r$ matrix will be nonsingular if and only if for any $q \in \mathbb{R}^r \setminus \{0\}$, $\psi = \sum_{i=1}^{r} q_i \psi_i(a)$, the quantity

$$-\langle \psi v_0, \psi v_0 \rangle + \langle v_0, \psi v_0 \rangle^2$$

does not vanish. But ψ is simply an arbitrary non zero element of $\underline{c}(a)$. Write $\psi = \psi_0 + \psi_1$, with $\psi_0 \in \underline{c}$ and $\psi_1 \in (\underline{n}_- \oplus \underline{n}_+) \cap \underline{h}^\perp$. Since $a \in \underline{h}^\perp \cap R \cap P$, the hypothesis [T] is satisfied at a; which means among other things that $\underline{c}(a) \cap \underline{c} = \{0\}$, hence $\psi_1 \neq 0$. It can be shown that, since $a \notin \underline{c}$, for suitable choice of ζ the subspace V_ζ is not an eigenspace of a. This implies that

$$0 \neq \psi_1 \cdot v_0 \in \sum_{\substack{\mu \in \Phi \\ \mu < \zeta}} V_\mu = V_\zeta^\perp .$$

Thus

$$-\langle \psi v_0, \psi v_0 \rangle + \langle v_0, \psi v_0 \rangle^2 = -\langle \psi_1 v_0, \psi_1 v_0 \rangle \neq 0$$

which completes the proof.

q.e.d.

Cor. 5.12. Suppose that $a \in \underline{h}^\perp \cap R \cap P$ lies on a completely polarized orbit $\mathcal{O}_a^G$ i.e. is a G-L point. Then the functions $\beta_1 \ldots \beta_r, \lambda_1, \ldots \lambda_r$ form a coordinate system on $\mathcal{O}_a^G$ in a neighborhood of a.

Proof. Suppose there were some $(q,p) \in \mathbb{R}^{2r}$ so that

$$\sum_{i=1}^{r} q_i d\beta_i(a) + p_i d\lambda_i(a) \Big|_{T_a \mathcal{O}_a^G} = 0 .$$

Then, for $k = 1, \ldots, r$

$$\sum_{i=1}^{r} q_i \{\beta_i, \lambda_k\}(a) = 0 .$$

Since $\{\lambda_i, \lambda_j\} \equiv 0$ thanks to Lemma 5.5 and Theorem 2.2, by the previous Proposition, $q = 0$. However the differentials of the λ's are everywhere independent (Lemma 5.6) so $p = 0$. Thus the $2r$ functions $\beta_1 \ldots \beta_r \; \lambda_1 \ldots \lambda_r$ when restricted to $\mathcal{O}_a^G$ have linearly independent differentials at a, hence define a coordinate system in some neighborhood of a, since $\dim \mathcal{O}_a^G = 2r$.

q.e.d.

Appendix to Section 5. Canonical Euclidean structure on highest weight modules $V(\lambda)$.

As in §5, $\underline{m}$ is a complex semisimple Lie Algebra, $\underline{\ell}$ a normal real form, $\underline{j} \subset \underline{\ell}$ a cartan subalgebra, Δ the set of roots for $(\underline{\ell},\underline{j})$ (i.e. the set of roots for $(\underline{m}, \underline{j} \otimes_{\mathbb{R}} \mathbb{C})$), Δ^+ a positive root system, $\underline{n}_+$ $(\underline{n}_-)$ the sum of the positive (negative) root spaces, $\underline{b} = \underline{j} \oplus \underline{n}_+$ the corresponding Borel sulalgebra.

Following Humphries, ([23] 20,21) for $\lambda \in \underline{j}^*$ construct the one-dimensional $\underline{b}$-module $D(\lambda) = \mathbb{R} \cdot v_0$ by the rule

$$(Z+x)v_0 = \lambda(z)v_0 \quad , Z \in \underline{j},\ x \in \underline{n}_+ \quad .$$

Denote by $\underline{U}(\underline{\ell})$ and $\underline{U}(\underline{b})$ the universal enveloping algebras (over $\mathbb{R}$) of $\underline{\ell}$ and $\underline{b}$. Then $D(\lambda)$ is naturally a $\underline{U}(\underline{b})$-module. Define (viewing $\underline{U}(\underline{\ell})$ as a right $\underline{U}(\underline{b})$-module)

$$Z(\lambda) = \underline{U}(\underline{\ell}) \otimes_{\underline{U}(\underline{b})} D(\lambda) \quad .$$

$Z(\lambda)$ is naturally a left $\underline{U}(\underline{\ell})$ module. According to the Poincaré-Birkhoff-Witt theorem, $\underline{U}(\underline{\ell})$ is a free $\underline{U}(\underline{b})$ module.

For this appendix, set for $\alpha \in \Delta^+$

$$x_\alpha = Z_\alpha \quad , \quad y_\alpha = Z_{-\alpha}$$

where $\{Z_\alpha\}$ is a Weyl basis. Also write

$$[x_\alpha, y_\alpha] = e_\alpha \in \underline{j}$$

so that $\{e_\alpha, x_\alpha, y_\alpha\}$ span a TDS of $\underline{\ell}$ for each $\alpha \in \Delta^+$.

Then a basis for $Z(\lambda)$ is

$$\{y_{\beta_1}^{i_1} \cdots y_{\beta_m}^{i_m} \otimes v_0 : m, i_1, \ldots, i_m \in \mathbb{Z}^+,\ \beta_1 \ldots \beta_m \in \Delta^+\} \quad .$$

From now on write v_0 for $1 \otimes v_0 \in Z(\lambda)$, so that you can omit the $\otimes$ in the above expression for basis vectors.

The module $Z(\lambda)$ is standard cyclic in the terminology of [23], and has the following properties:

1) $Z(\lambda) = \underline{U}(\underline{n}_-) \cdot v_0 = \underline{U}(\underline{\ell}) \cdot v_0$

2) $Z(\lambda) = \bigoplus_{\mu \leq \lambda} Z(\lambda)_\mu$

where $<$ is the usual partial order in $\underline{j}^*$ and

$$Z(\lambda)_\mu = \{v \in Z(\lambda) : z \cdot v = \mu(z)v \ \forall z \in \underline{j}\} .$$

The $\mu \in \underline{j}^*$ for which $Z(\lambda)_\mu \neq 0$ have the form

$$\mu = \lambda - \sum_{\alpha \in \Delta^+} k_\alpha \alpha, \quad k_\alpha \in \mathbb{Z} .$$

If μ has this form, a basis for $Z(\lambda)_\mu$ is as follows. Let the positive roots be described in a fixed order $\Delta^+ = \{\alpha_1 \cdots \alpha_m\}$. For $I = (I_1 \cdots I_m) \in \mathbb{Z}^m$, set

$$v_I = y_{\alpha_1}^{I_1} \cdots y_{\alpha_m}^{I_m} v_0 \quad y^I \cdot v_0 .$$

For a dominant integral weight ν, let $P(\nu)$ denote the collection of $I \in \mathbb{Z}^{+m}$ for which $\nu = \sum_{i=1}^{m} I_i \alpha_i \equiv I \cdot \alpha$. Then a basis for $Z(\lambda)_\mu$ is

$$\{v_I : I \in P(\lambda - \mu)\} .$$

In particular, $\dim Z(\lambda)_\mu = |P(\lambda-\mu)| < \infty$.

3). $Z(\lambda)$ is an indecomposable $\underline{\ell}$-module, and has a unique maximal proper submodule, described as follows: Set

$$x^I = x_{\alpha_1}^{I_1} \cdots x_{\alpha_m}^{I_m} \quad \text{for } I \in \mathbb{Z}^{+m} .$$

Note that, if $v \in Z(\lambda)_\mu$, then

$$y^I \cdot v \in Z(\lambda)_{\mu - I \cdot \alpha}$$

$$x^I \cdot v \in Z(\lambda)_{\mu + I \cdot \alpha}$$

so $x^I \cdot v \in Z(\lambda)_\lambda = \mathbb{R} \cdot v_0$ if $I \in P(\lambda-\mu)$, and $x^I \cdot v = 0$ if $I \cdot \alpha > \lambda - \mu$.

Set $$Y(\lambda)_\mu = \{v \in Z(\lambda)_\mu : x^I \cdot v = 0 \ \ \forall I \in P(\lambda-\mu)\}$$

$$Y(\lambda) = \bigoplus_{\mu<\lambda} Y(\lambda)_\mu \ .$$

Then $Y(\lambda)$ is the unique maximal proper sub module of $Z(\lambda)$.

The main theorem in the subject is as follows. Let $\alpha_1,\ldots,\alpha_r$ be the simple roots of the positive system Δ^+ (see [23] 10.1), and put $z_i = [x_i,y_i]$. Then $Z_i \in \underline{j}$, and x_i, y_i, z_i span a subalgebra of ℓ isomorphic to $s\ell(2\ \mathbb{R})$. A weight $\lambda \in \underline{j}^*$ is called <u>integral</u> if $\lambda(z_i) \in \mathbb{Z}$, $i = 1\cdots\ell$, and dominant if $\lambda(z_i) \geq 0$, $i = 1\cdots\ell$.

The theorem alluded to is: $V(\lambda) = Z(\lambda)/Y(\lambda)$ is a finite-dimensional irreducible $\underline{\ell}$-module if and only if λ is dominant integral, and any irreducible finite-dimensional $\underline{\ell}$-module is isomorphic to one of these (see [23], 21.]).

Now define a bilinear form B on $Z(\lambda)$ by the requirements

1) $B(v_0,v_0) = 1$

2) $B(Z(\lambda)_\mu,\ Z(\lambda)_\nu) = 0, \quad \mu \neq \nu$

3) For $I,\ I' \in P(\lambda-\mu)$,

$$B(v_I,v_{I'}) = B(\widehat{x^{I'}}\, v_I,v_0)$$

where $\widehat{x^I} = x_m^{I_m}\cdots x_1^{I_1}$.

<u>Theorem</u> (A1)

1) B is symmetric

2) $\ker B \cap Z_\mu(\lambda)_\mu = Y(\lambda)_\mu$

3) For $y \in \underline{n}_-$, $B(v,y\cdot w) = B(x\cdot v,w)$, $x = \sigma(y)$.

<u>Proof</u>: Suppose $z \in \underline{j}$. Then

$$z \cdot \widehat{x^I} v_J = z \cdot \widehat{x^I} y^J v_0$$

$$= (\lambda + I\cdot\alpha - J\cdot\alpha)(z) v_0 \ .$$

According to fact 2) above, this is zero unless $(J-I)\cdot\alpha \geq 0$, since otherwise $\lambda + (I-J)\cdot\alpha$

is not a weight of $Z(\lambda)$.

Suppose $I \in P(\lambda-\mu)$, $J \in P(\lambda-\nu)$. Then either

(i) $(J-I)\cdot\alpha > 0$, whence $\widehat{x^I}v_J \in Z(\lambda)_\phi$ with $.\ \phi = \lambda + (I-J)\alpha < \lambda$ and $\therefore$

$$B(\widehat{x^I}v_J, v_0) = 0 \quad [\text{by 2) since } \mu \neq \nu]$$

$$= B(v_J, v_I)$$

(ii) $(J-I)\cdot\alpha = 0$, i.e. $\mu = \nu$

or

(iii) $(J-I)\cdot\alpha \neq 0$, so $\mu \geq \nu$ and both $B(\widehat{x^I}v_J, v_0) = 0 = B(v_J, v_I)$ since $\widehat{x^I}v_J = 0$.

So in any case the formula

$$B(v_J, v_I) = B(\widehat{x^I}v_J, v_0)$$

holds for all $I, J \in \mathbb{Z}^{+m}$. By linearity,

$$B(w, v_I) = B(\widehat{x^I}w, v_0), \quad w \in Z(\lambda) \quad .$$

Next you establish 3) as follows: Note that σ extends to a linear automorphism of $\underline{U}(\underline{\ell})$, with the property that

$$\sigma(zw) = \sigma(w)\sigma(z) \quad .$$

So for instance $\sigma(y^I) = \widehat{x^I}$. For $y \in \underline{n}_-$, $I \in \mathbb{Z}^{+n}$ you can write

$$y \cdot y_I = \sum_{m \leq n+1} \sum_{J \in \mathbb{Z}^{+m}} c_J(I) y^J$$

where only finitely many $c_J(I) \neq 0$, so

$$B(v_{I'}, yv_I) = B(v_{I'}, yy_I v_0)$$

$$= B(v_{I'}, \sum_{m \leq n+1} \sum_{J \in \mathbb{Z}^{+m}} c_J(I) v_J)$$

$$= \sum_{m \leq n+1} \sum_{J \in \mathbb{Z}^{+m}} c_J(I) B(v_{I'}, v_J)$$

$$= \sum_{m \le n+1} \sum_{J \in \mathbb{Z}^{+m}} c_J(I) B(\widehat{x^J} v_{I'}, v_0)$$

$$= B\Big(\Big(\sum_{m \le n+1} \sum_{J \in \mathbb{Z}^{+m}} c_J(I) \widehat{x^J}\Big) \cdot v_{I'}, v_0\Big)$$

$$= B\Big(\sigma\Big(\sum_{m \le n+1} \sum_{J \in \mathbb{Z}^{+m}} c_J(I) y^J\Big) v_{I'}, v_0\Big)$$

$$= B(\sigma(yy^I) \cdot v_{I'}, v_0)$$

$$= B(\widehat{x^I} \cdot x \cdot v_{I'}, v_0)$$

$$= B(x \cdot v_{I'}, v_I) \quad .$$

Since the v_I are a basis and B is bilinear, 3) is established.

As above, let $\alpha_1, \ldots, \alpha_r$ be the set of simple roots. Since the simple root vectors $x_1, \ldots, x_r$ generate $\underline{n}_+$, respectively $y_1, \ldots, y_r$ generate $\underline{n}_-$, you can write any v_I, hence any element of $Z(\lambda)$, as a sum of elements of the form

$$y_{i_1} \cdots y_{i_k} \cdot v_0 \quad , \quad 1 \le i_1, \ldots, i_k \le r$$

(repetitions allowed).

You have proved the symmetry part of 1), therefore, if you show that

$$\begin{aligned} &B(y_{i_1} \cdots y_{i_k} v_0, y_{j_1} \cdots y_{j_\ell} v_0) \\ &\quad = B(y_{j_1} \cdots y_{j} v_0, y_{i_1} \cdots y_{i_k} v_0) \end{aligned} \qquad \text{(I)}$$

for all sequences i, j in $\{1, \ldots r\}$.

The proof is a double induction on the <u>length</u> $k + \ell$ of the sequence (i, j). Simultaneously with (I) consider the assertion

$$\begin{aligned} &B(y_{i_1} \cdots y_{i_k} v_0, x_p y_{j_1} \cdots y_{j_\ell} v_0) \\ &\quad = B(y_p y_{i_1} \cdots y_{i_k} v_0, y_{j_1} \cdots y_{j_\ell} v_0) \quad . \end{aligned} \qquad \text{(II)}$$

Attach subscripts to denote the length for which the identities are asserted: thus the above

are $I_{k+\ell}$, $II_{k+\ell}$.

Note that II_0 is

$$B(v_0, x_p v_0) = B(y_p v_0, v_0) = 0 \quad .$$

For $k + \ell = 0$, I_0 is part 1) of the definition of B. For $k + \ell = 1$,

$$B(v_0, y_j v_0) = B(y_j v_0, v_0) = 0$$

by 2), since $y_j \cdot v_0 \in Z(\lambda)_{\lambda-\alpha_j}$.

For $k + \ell \geq 2$, refer the symmetry property $(I_{k+\ell})$ to the same for length $k + \ell - 2$ as follows: Recall that $[x_j, y_{j'}] = 0$, $j \neq j'$, since $\alpha_j - \alpha_{j'}$ is not a root if $\alpha_j, \alpha_{j'}$ are simple.

$$B(y_{i_1} \cdots y_{i_k} v_0, y_{j_1} \cdots y_{j_\ell} v_0)$$

$$= B(x_{j_1} y_{i_1} \cdots y_{i_k} \cdot v_0, y_{j_2} \cdots y_{j_\ell} \cdot v_0) \qquad \text{(by 3)}$$

$$= \sum_{p=1}^{k} \delta_{j_1, i_p} (\lambda \alpha_{i_{p+1}} \cdots \alpha_{i_k})(Z_{j_1})$$

$$\times B(y_{i_1} \cdots \widehat{y_{i_p}} \cdots y_{i_k} \cdot v_0, y_{j_2} \cdots y_{j_\ell} \cdot v_0)$$

where here the hat means deletion. By induction $(I_{k+\ell-2})$

$$= B(y_{j_2} \cdots y_{j_\ell} \cdot v_0, x_{j_1} y_{i_1} \cdots y_{i_k} \cdot v_0) \quad .$$

Thanks to $II_{k+\ell-1}$, this is

$$= B(y_{j_1} \cdots y_{j_\ell} \cdot v_0, y_{i_1} \cdots y_{i_k} \cdot v_0)$$

so $I_{k+\ell-2}$ and $II_{k+\ell-1}$ imply $I_{k+\ell}$. On the other hand,

$$B(y_{i_1}\cdots y_{i_k}v_0,\ x_p y_{j_1}\cdots y_{j_\ell}v_0)$$

$$= \sum_{q=1}^{\ell} \delta_{p,j_q}(\lambda-\alpha_{j_{q+1}}-\cdots-\alpha_{j_\ell})(z_p)$$

$$\times B(y_{i_1}\cdots y_{i_k}\cdot v_0,\ y_{j_1}\cdots \widehat{y_{j_q}}\cdots y_{j_\ell}\cdot v_0)$$

so an application of $I_{k+\ell-1}$ establishes $II_{k+\ell}$. The induction is complete.

Finally 2) of the Theorem is an obvious consequence of 3) of the definition.

q.e.d.

Conclusion 2) allows you to view B as a bilinear form on $V(\lambda) = Z(\lambda) / Y(\lambda)$.

<u>Lemma A2</u>. Let $\underline{\ell} = s\ell(2,\mathbb{R})$, and suppose that λ is a dominant integral weight, so that $V(\lambda)$ is finite-dimensional. Then B is positive-definite on $V(\lambda)$.

<u>Proof</u>: Let x, y, z be the usual basis of $s\ell(2,\mathbb{R})$, so that

$$[x,z] = -2x,\ [y,z] = 2y,\ [x,y] = z\ .$$

Set $m = \zeta(z) \in \mathbb{Z}^+$, and write

$$v_k = \frac{1}{k!}\, y^k\cdot v_0\ .$$

Then as in [23] §7.2

$$z \cdot v_k = (m-2k)v_k$$

$$y \cdot v_k = (k+1)v_{k+1}$$

$$x \cdot v_k = (m-k+1)v_{k-1}\ .$$

The vectors $v_0,\ldots,v_m$ span $V(\zeta)$. To show B positive-definite, we need only show that $B(v_k,v_k) > 0$. This is <u>ex. def.</u> true for $k = 0$. Suppose that it is true for $v_0,\ldots,v_k$. Then

$$B(v_{k+1},v_{k+1}) = \frac{1}{(k+1)}\, B(v_{k+1},yv_k)$$

$$= \frac{1}{k+1}\, B(xv_{k+1},v_k)$$

$$= \frac{m-k}{k+1}\, B(v_k,v_k) > 0$$

for $k = 0, \quad m-1$.

q.e.d

The idea of the proof of positive-definiteness in general is to use the existence of lots of subalgebras of $\mathfrak{l}$ isomorphic to $s\ell(2,\mathbb{R})$.

Theorem A.3. Suppose λ is dominant integral. Then B is positive-definite.

Proof: Suppose not. Then there is a weight μ of minimal length $\ell = \ell(\mu)$ so that $B|_{V_\mu}$ is not positive-definite. (The definition of the length of a weight follows the proof of Prop. 6.4 in the next section.) Select $0 \neq v \in V_\mu$ for which $B(v,v) \leq 0$. Since $v \neq 0$, there must exist $\alpha \in \Delta$ for which $x_\alpha \cdot v \neq 0$. (Indeed, this is a property of all vectors which survive in the quotient $V = Z/Y$, as is evident from the definition of Y given above.)

Let $\mathcal{S}$ denote the span of x_α, y_α, and $z_\alpha = [x_\alpha, y_\alpha]$. Then $\mathcal{S}$ is a subalgebra of $\mathfrak{l}$ isomorphic to $s\ell(2,\mathbb{R})$. Let V' be the cyclic $\mathcal{S}$-module generated by v. V' is indecomposable, since it is cyclic; since $\mathcal{S}$ is semisimple, V' is actually irreducible (Weyl's Theorem). Let v_+ be a maximal vector for V', and let $v_0 = v_+, v_1, \ldots, v_m$ be a basis of weight vectors as in Lemma A.2. You have

$$v = \sum_{k=0}^{m} c_k v_k$$

$$z_\alpha \cdot v = \mu(z_\alpha) v = \sum_{k=0}^{m} c_k (m-2k) v_k$$

$$\text{i.e.} \quad \mu(z_\alpha) c_k = (m-2k) c_k, \; k = 0, \ldots, m$$

hence for some k, $\mu(z_\alpha) = m = 2\,k$ and $v = cv_k$, $c \neq 0$. It follows from the commutation relations in the proof of Lemma A.2 that v_+ is proportional to $x_\alpha^k \cdot v \neq 0$ and is in particular a weight vector of $V(\lambda)$. Since $x_\alpha \cdot v \neq 0$, v_+ has weight strictly higher than μ. It is clear from the constructions, however, that $B|_{V'}$ is proportional to the form on V' constructed as in Lemma A.2, which is positive definite, therefore $B|_{V'}$ is either positive-definite, negative-definite, or identically zero. Since μ was the weight of minimal length for which B is not > 0 on V_μ, however, and v_+ has weight $> \mu$, hence length $< \ell(\mu)$, it follows that $B(v_+, v_+) > 0$, which contradicts the hypothesis that $B(v,v) \leq 0$.

q.e.d.

§6. Solution of the Equations of Motion

The notation of this section continues that of the last. In particular, $a_0 \in \underline{h}^{\perp}$ is some reference point, and U is a neighborhood of a_0 having the properties stated after Lemma 5.9.

The first step is a result similar to Prop. 5.11, for which you first need to compute some derivatives:

Lemma 6.1

(i) $d\rho_{\mu}(a,[z,a]) = \langle v_0,[z,P_{\mu}(a)]v_0\rangle$ for $a \in U$, $z \in \underline{h}$, $\mu \in \Phi$

(ii) $\nabla\mu^*(a) = c_{\mu}(a) \equiv \mathrm{Ad}(Y^{-1}(a,a_0)Y_0^{-1})(c_{\mu})$.

Proof:

(i) $$d\rho_{\mu}(a,[z,a]) = \frac{d}{dt}\Big|_{t=0} \langle v_0, P_{\mu}(e^{t\cdot \mathrm{ad} z}(a))v_0\rangle$$

$$= \frac{d}{dt}\Big|_{t=0} \langle v_0\ e^{t\cdot \mathrm{ad} z} P_{\mu}(a)v_0\rangle$$

$$= \langle v_0,[z,P_{\mu}(a)]v_0\rangle \ .$$

(ii) Note that $\mu^*(a) = K(c_{\mu}(a),a)$ then ape the proof of Lemma 5.6.

q.e.d.

Proposition 6.2 for $a \in U$, $\mu, \nu \in \Phi$,

$$\{\rho_{\mu},\nu^*\}(a) = -2 \sum_{\mu' \in \Phi} K^*(\nu,\mu')(\delta_{\mu\mu'}\rho_{\mu} - \rho_{\mu}\rho_{\mu'})$$

where K^* is the form on $\underline{c}^* \times \underline{c}^*$ dual to K; $K^*(\mu,\nu) = K(c_{\mu},c_{\nu}) = \mu(c_{\nu})$, and $\delta_{\mu\mu} = 1$, $\delta_{\mu\mu'} = 0$, $\mu \neq \mu' \in \Phi$.

Proof: $\{\rho_{\mu},\nu^*\}(a) = K(a, [\Pi_{\underline{g}}\nabla\rho_{\mu}(a),\Pi_{\underline{g}}\nabla\nu^*(a)])$

$= -K(\Pi_{\underline{g}}\nabla\rho_{\mu}(a), [a, \Pi_{\underline{g}}\nabla\nu^*(a)])$

$= K(\Pi_{\underline{g}}\nabla\rho_{\mu}(a), [a, \Pi_{\underline{h}}\nabla\nu^*(a)])$

$$= K(\nabla\rho_\mu(a),\ [a,\ \Pi_{\underline{h}}\nabla\nu^*(a)])$$

$$= d\rho_\mu(a,\ [a,\ \Pi_{\underline{h}}\nabla\nu^*(a)])$$

$$= \langle v_0,\ [P_\mu(a),\ \Pi_{\underline{h}} c_\nu(a)], v_0\rangle$$

$$= 2\langle P_\mu(a)v_0,\ \Pi_{\underline{h}} c_\nu(a)v_0\rangle$$

$$= -2\langle P_\mu(a)v_0,\ c_\nu(a)v_0\rangle$$

$$+2\langle v_0,\ P_\mu(a)v_0\rangle\langle v_0,\ c_\nu(a)v_0\rangle$$

as in the proof of Prop. 5.11. Now

$$c_\nu = \sum_{\mu'\in\Phi} \mu'(c_\nu)P_{\mu'}$$

$$= \sum_{\mu'\in\Phi} K(\mu',\nu)P_{\mu'} \quad .$$

Also, since the P's are orthogonal projections,

$$\langle P_\mu(a)v_0,\ P_{\mu'}(a)v_0\rangle = \delta_{\mu\mu'}\langle v_0,\ P_\mu(a)v_0\rangle$$

since

$$\langle v_0,\ P_\mu(a)v_0\rangle = \rho_\mu(a) \quad ,$$

these remarks suffice to complete the proof.

q.e.d.

For $a_0 \in \underline{h}^\perp$, U selected as in the last section, denote by N the integral manifold of the vector fields $\{D_F : F \in S(U)\}$ through a_0.

Lemma 6.3. Suppose that the neighborhood $U \cap N$ of a_0 in N, is connected and simply connected. Then for each $\tau_0 \in \mathbb{R}$ there exists a unique function $\tau : U \cap N \to \mathbb{R}$ so that for all $\nu \in \Phi$

$$\{\tau,\nu^*\} = -2\sum_{\mu'\in\Phi} K(\nu,\mu')\rho_{\mu'},\quad \tau(a_0) = \tau_0 \quad .$$

Proof: This follows easily from the following fact, which is in turn a consequence of the Poincaré Lemma on exactness of closed 1-forms:

Suppose U is a connected, simply connected smooth manifold, $D_1,\ldots,D_n$ smooth vector fields which span $T_x U$ for all $x \in U$, $f_1,\ldots,f_k$ smooth functions on U and $x_0 \in U$. Then the problem

$$u(x_0) = u_0, \; D_j u = f_j, \; j = 1,\ldots,n$$

has a unique smooth solution in U if and only if the following integrability conditions are satisfied:

Let $\omega_{ij}^k(x)$ be any functions on U so that $D_i D_j - D_j D_i = \sum_{k=1}^n \omega_{ij}^k D_k$. Then

$$D_i f_j - D_j f_i = \sum_{k=1}^n \omega_{ij}^k f_k \quad .$$

There are always enough weights in any irreducible finite-dimensional representation to span $\underline{c}^*$, so (reasoning as in Lemma 5.5), the functions $\{\nu^* : \nu \in \Phi\}$ generate the Ad-invariant polynomials, and therefore the Hamiltonian vector fields D_{ν^*} span TN. To apply the principle just enunciated, compute

$$D_{\nu_1^*} \cdot \Big(-2 \sum_{\mu' \in \Phi} K(\nu_2,\mu')\rho_{\mu'}\Big)$$

$$= -2 \sum_{\mu' \in \Phi} K(\nu_2,\mu')\{\rho_{\mu'},\nu_1^*\}$$

$$= 4 \sum_{\mu',\mu'' \in \Phi} K(\nu_1,\mu'')K(\nu_2,\mu')(\delta_{\mu'\mu''}\rho_{\mu'} - \rho_{\mu'}\rho_{\mu''}) \quad .$$

This expression is symmetric in $\nu_1,\nu_2 \in \Phi$, and the vector fields D_{ν^*} commute (Thm. 2.2) so the above principle guarantees the existence and uniqueness of the required solution τ.

q.e.d.

Proposition 6.4. Let $a_\nu(t)$ be the trajectory of the Hamiltonian vector field D_{ν^*}, $\nu \in \Phi$, such that $a_0 = a_\nu(0)$. Then

$$\mu^*(a_\nu(t)) = \mu^*(a_0)$$

$$\rho_\mu(a_\nu(t)) = \frac{e^{-2K^*(\mu,\nu)t}\rho_\mu(a_0)}{\sum\limits_{\mu'\in\Phi} e^{-2K^*(\mu',\nu)t}\rho_{\mu'}(a_0)}$$

for all $\mu \in \Phi$.

Proof: The first statement is implied by Thm. 2.2. To see the second statement, let τ be as in Lemma 6.3, with $\tau(a_0) = 1$, and set

$$\chi_\mu(a) = e^{\tau(a)}\rho_\mu(a) \quad .$$

Then

$$\begin{aligned}\{\chi_\mu,\nu^*\} &= \{e^\tau \nu^*\}\rho_\mu + \{\rho_\mu,\nu^*\}e^\tau \\ &= \{\tau,\nu^*\}\chi_\mu + \{\rho_\mu,\nu^*\}e^\tau \\ &= -2\sum_{\mu'\in\Phi} K^*(\nu,\mu')\rho_{\mu'}\rho_\mu e^\tau + 2e^\tau \sum_{\mu'\in\Phi} K^*(\mu',\nu)(\rho_\mu\rho_{\mu'} - \delta_{\mu\mu'}\rho_\mu) \\ &= -2K(\mu,\nu)\chi_\mu \quad .\end{aligned}$$

Therefore
$$\begin{aligned}\chi_\mu(a_\nu(t)) &= e^{-2K(\mu,\nu)t}\chi_\mu(a_0) \\ &= e^{-2K(\mu,\nu)t}\rho_\mu(a_0) \quad .\end{aligned}$$

Now from the definition of ρ_μ you see that

$$\sum_{\mu\in\Phi}\rho_\mu \equiv 1$$

so $\sum\limits_{\mu\in\Phi}\chi_\mu = e^\tau$. Since $\rho_\mu = e^{-\tau}\chi_\mu$, the asserted formula follows immediately.

q.e.d.

This result may also be written

$$r_\mu(a_\nu(t)) = \frac{e^{-K^*(\mu,\nu)t}\, r_\mu(a_0)}{\sqrt{\sum_{\mu'\in\Phi} e^{-2K^*(\mu',\nu)t}\, r^2_{\mu'}(a_0)}}$$

(compare Moser [2]).

The same proof shows;

Cor. 6.5. For any Ad-invariant function F on $\underline{h}^\perp$, let $a_F(t)$ be the trajectory of the Hamiltonian vector field $\mathcal{D}_F$ with initial condition $a_F(0) = a_0$. Then

$$\rho_\mu(a_F(t)) = e^{-2_\mu(\nabla_{\underline{c}} F(a))t}\, \rho_\mu(a_0) \left(\sum_{\mu'\in\Phi} e^{-2\mu'(\nabla_{\underline{c}} F(a))\cdot t}\, \rho_{\mu'}(a_0) \right)^{-1}$$

where

$$\nabla_{\underline{c}} F(a) = \mathrm{Ad}(Y_0^{-1} Y(a,a_0))(\nabla F(a))$$

(which is itself an ad-invariant $\underline{c}$-valued function on $\mathcal{U}$).

Expecially, if F is a homogeneous trace polynomial

$$F(a) = \frac{1}{k} \operatorname{tr} a^K$$

(here a is viewed as an endomorphism of V) then

$$\rho_\mu(a_F(t)) = e^{-2(\mu^*(a))^{k-1}\cdot t}\, \rho_\mu(a_0) \left(\sum_{\mu'\in\Phi} e^{-2(\mu'^*(a))^{k-1}\cdot t}\, \rho_{\mu'}(a_0) \right)^{-1} .$$

Remark. It can be shown ([23] §23.1) that the trace polynomials generate the ring of Ad-invariants of $\underline{\ell}$.

The final point of our work is that the matrix elements of trajectories of Ad-invariant Hamiltonian, on completely integrable orbits, are actually rational functions of linear exponentials, with coefficients rationally determined by the initial data and the spectrum $\{\mu^*\}$. First you need several facts about the action of G on V.

Recall (Appendix to §5) that the weights of V are all of the form $\mu = \zeta - \sum_{\alpha\in S} k_\alpha \alpha$, S the fixed simple roots of $\underline{\ell}$. Set

$$\ell(\mu) = \sum_{\alpha\in S} k_\alpha \in \mathbb{Z}$$

and call $\ell(\mu)$ the length of μ.

Some notation for the various "matrix blocks" will also be useful. For $X \in \operatorname{End} V$, $\mu, \mu' \in \Phi$, set

$$X_{\mu\mu'} = P_{\mu'} X P_{\mu} \ .$$

Lemma 6.6. Suppose $x \in \underline{g}$. Then

$$x_{\mu\mu'} = 0$$

if $\ell(\mu') < \ell(\mu)$ or $\ell(\mu') = \ell(\mu)$, $\mu' \neq \mu$. Also, $x_{\mu\mu}$ is proportional to the identity on V_{μ}.

Proof: Write

$$x = \sum_{\alpha \in \Delta} x_{\alpha} Z_{-\alpha} + x_i c_i$$

with $\{Z_{-\alpha}, Z_{\alpha}, c_i\}$ a Weyl basis of $\underline{\ell}$. Then for $v \in V_{\mu}$, $\mu \in \Phi$,

$$x \cdot v = \sum_{\alpha \in \Delta} x_{\alpha} Z_{-\alpha} v + x_i c_i \cdot v$$

$$\in \bigoplus_{\ell(\mu') > \ell(\mu)} V_{\mu'} \oplus V_{\mu}$$

since $Z_{-\alpha} \cdot V_{\mu} \subset V_{\mu - \alpha}$, etc.

q.e.d.

Cor. 6.7. The connected group G is an exponential group.

Proof: Order the weights according to length, and order the weights of a given length anyhow. Pick a basis of weight vectors. The above result shows that $\underline{g}$ is a subalgebra of the algebra of lower triangular matrices $\underline{t}$ in this basis. Since the group T of lower triangular entries with positive diagonal entries is exponential with Lie algebra $\underline{t}$, so is G exponential.

q.e.d.

Cor. 6.8. For $X \in G$, $X_{\mu\mu'} = 0$ if $\ell(\mu') < \ell(\mu)$ or $\ell(\mu') = \ell(\mu)$, $\mu' \neq \mu$. Moreover, $X_{\mu\mu}$ is a positive multiple of the identity on V_{μ}.

Proof: Since the analogous property for $\underline{g}$ is stable under formation of powers and G is exponential, this is obvious.

q.e.d.

The next step is to examine the "isospectral" sets $\mathcal{O}^L \cap \mathcal{O}^G$, when $\mathcal{O}^G$ is completely polarized. From now on suppose that a_0 is a $G-L$ point. Then Theorem 4.5 asserts the existence of a neighborhood $\mathcal{U}$ of a_0 so that for $a \in \mathcal{U} \cap \mathcal{O}^L_{a_0} \cap \mathcal{O}^G_{a_0}$, there are $Y \in H$ and $X \in G$ so that

$$a_0 = \mathrm{Ad}\, X(a) = \mathrm{Ad}\, Y(a)$$

or (as endomorphisms of V)

$$a_0 = X\, a\, X^{-1} = Y\, a\, Y^{+} \quad .$$

Especially, $X^{-1}Y$ is in the Ad-isotropy subgroup of a. If $\mathcal{U}$ is chosen small enough, $X^{-1}Y$ will actually be the exponential of an element of $\underline{c}(a)$, hence expressible (as remarked in §5) in the form

$$X^{-1}Y = \sum_{\mu \in \Phi} \kappa_\mu P_\mu(a)$$

for positive constants κ_μ. Thus

$$Y = \sum_{\mu \in \Phi} \kappa_\mu\, X\, P_\mu(a) \quad .$$

Now

$$P_\mu(a_0) = Y\, P_\mu(a) Y^{+}$$

according to the definitions of §5

$$= \Big(\sum_{\mu' \in \Phi} \kappa_{\mu'}\, X\, P_{\mu'}(a)\Big) P_\mu(a_0) \Big(\sum_{\mu'' \in \Phi} \kappa_{\mu''} P_{\mu''}(a) X^{+}\Big)$$

$$= \kappa_\mu^2\, X\, P_\mu(a) X^{+}$$

or

$$\kappa_\mu^{-2} P_\mu(a_0) = X\, P_\mu(a) X^{+} \quad .$$

Finally, since $\sum_{\mu \in \Phi} P_\mu = 1$,

$$\sum_{\mu\in\Phi} \kappa_\mu^{-2} P_\mu(a_0) = XX^+ \qquad \text{(G-L)}$$

__Remark__. This equation will be the key to the rational expression of matrix elements in terms of the functions ρ_μ. The (G-L) stands for Gel'fand-Levitan, for reasons explained in the introduction.

__Theorem__ 6.9. Suppose that $Y \in H$,

$$\bar{P}_\mu = Y P_\mu(a_0) Y^+ \qquad \mu \in \Phi \quad ,$$

and $\sigma_\mu > 0$, $\mu \in \Phi$. Then the equation

$$M \equiv \sum \sigma_\mu \bar{P}_\mu = XX^+$$

has at most one solution $X \in \text{End } V$ with the properties

(i) $X_{\mu\mu} = d_\mu \cdot P_\mu$, $d_\mu > 0$,

(ii) $X_{\mu\mu'} = 0$ *if* $\ell(\mu') < \ell(\mu)$ *or* $\ell(\mu') = \ell(\mu)$, $\mu' \neq \mu$.

If a solution exists, it is invertible, and its matrix elements in any basis are rational functions of the $\{\sigma_\mu\}$ and the matrix elements of $\{P_\mu(a_0)\}$.

__Proof__: The proof will be by induction on $\ell(\mu)$.

1. $\ell(\mu) = 0$ (i.e. $\mu = \zeta$).

Note that $(XX^+)_{\mu\nu} = \sum_{\mu'\in\Phi} X_{\mu'\nu} X^*_{\mu\mu'}$ and also $X^+_{\mu\mu'} = (X_{\mu'\mu})$. Hence

$$(XX^+)_{\zeta\zeta} = \sum_{\mu'\in\Phi} X_{\mu'\zeta} X^+_{\zeta\mu'}$$

$$= X_{\zeta\zeta} X^+_{\zeta\zeta} = d_\zeta^2 \cdot P_\zeta$$

according to properties (i) (ii) above (expecially, if $\mu \in \Phi$, $\mu \neq \zeta$, then $\ell(\mu) > \ell(\zeta)$.)

Since V_ζ is one-dimensional,

$$M_{\zeta\zeta} = m_\zeta \cdot P_\zeta \quad , \quad m_\zeta > 0 \quad .$$

Therefore $d_\zeta = +\sqrt{m_\zeta}$.

2. Suppose that, for μ, ν such that $\ell(\mu) < \ell > \ell(\nu)$,

$$M_{\mu\nu} = (XX^+)_{\mu\nu} \quad .$$

Thus $X_{\mu\nu}$ is determined for $\ell(\mu), \ell(\nu) < \ell$. Take $\mu \in \Phi$ with $\ell(\mu) = \ell$. Property (i) determines all $X_{\nu\mu}$ except those with $\ell(\nu) < \ell$ or $\nu = \mu$.

To determine these, use a subsidiary induction on $\ell(\nu)$. For $\ell(\nu) = 0$, i.e., $\nu = \zeta$,

$$\begin{aligned} M_{\zeta\mu} &= (XX^+)_{\zeta\mu} \\ &= \sum_\nu X_{\nu\mu} X^+_{\zeta\nu} = X_{\zeta\mu} X^+_{\zeta\zeta} \\ &= d_\zeta X_{\zeta\mu} \end{aligned}$$

so $X_{\zeta\mu} = d_\zeta^{-1} M_{\zeta\mu}$.

For $0 < \ell(\nu) < \ell$, assume that $X_{\mu'\mu}$ is known, for $0 \leq \ell(\mu') < \ell(\nu)$. Then

$$\begin{aligned} M_{\nu\mu} &= (XX^+)_{\nu\mu} \\ &= \sum_{\mu'} X_{\mu'\mu} (X^+)_{\nu\mu'} = \sum_{\mu'} X_{\mu'\mu} (X_{\mu'\nu})^+ \\ &= \sum_{\ell(\mu') < \ell(\nu)} X_{\mu'\mu} (X_{\mu'\nu})^+ + X_{\nu\mu} X_{\nu\nu} \quad . \end{aligned}$$

According to the main induction hypothesis, $X_{\mu'\nu}$ is known for all μ'; also $X_{\nu\nu} = d_\nu \cdot P_\nu$ is known. According to the subsidiary induction hypothesis $X_{\mu'\mu}$ is known for all μ' appearing in the sum. Therefore the equation may be solved:

$$X_{\nu\mu} = d_\nu^{-1} (M_{\mu\nu} - \sum_{\ell(\mu') < \ell(\nu)} X_{\mu'\mu} (X_{\mu'\nu})^+) \quad .$$

Finally, for $\nu = \mu$

$$M_{\mu\mu} = \sum_{\ell(\nu)<\ell(\mu)} X_{\nu\mu}(X_{\nu\mu})^{+} + X_{\mu\mu}(X_{\mu\mu})^{+} \quad .$$

According to the main induction, the sum on the R.H.S. is known. The equation has a solution if and only if

$$M_{\mu\mu} - \sum_{\ell(\nu)<\ell(\mu)} X_{\nu\mu}(X_{\nu\mu})^{+}$$

is a positive scalar multiple of P_μ, say $d_\mu^2 P_\mu$, in which case $X_{\mu\mu} = d_\mu P_\mu$.

Thus the $X_{\nu\mu}$, $\ell(\nu) < \ell(\mu)$, and $X_{\mu\mu}$ are unique and depend rationally on M, as required. The induction is therefore complete.

q.e.d.

Remark. The proof actually gives existence, provided that the weight spaces are all one-dimensional. The general existence question is open.

Before applying the theorem to the case at hand, identify the constants κ_μ as follows: since

$$\kappa_\mu^{-2} P_\mu(a_0) = X\, P_\mu(a) X^{+}$$

$$\langle v_0, \kappa_\mu^{-2} P_\mu(a_0) v_0\rangle = \langle \cdot X^{+} v_0, P_\mu(a) X^{+} v_0\rangle \quad .$$

Since $X \in G$, it follows from Cor. 6.8 that $X^{+} v_0 = d_\zeta v_0$, where

$$d_\zeta = \langle v_0, X^{+} v_0\rangle \quad .$$

Using the definition of $r_\mu(a)$ (§5) you see that

$$\kappa_\mu^{-2} v_\mu^2(a_0) = d_\zeta^2 v_\mu^2(a) \quad .$$

Summing both sides and using $\sum r_\mu^2 \equiv 1$, obtain

$$\sum \kappa_\mu^{-2} r_\mu^2(a_0) = d_\zeta^2 = (\kappa_\mu^{-1} r_\mu(a_0) r_\mu(a)^{-1})^2 \quad .$$

Now denote by Ω_{a_0} the collection of sets $\{\bar{r}_\mu^{-2} : \mu \in \Phi\}$ of numbers $r_\mu^{-2} > 0$ so that $\bar{r}_\mu = r_\mu(a)$ for some $a \in U \cap O_{a_0}^{G} \cap O_{a_0}^{L}$, all $\mu \in \Phi$.

Theorem 6.10. Suppose a_0 ia a G-L point, so that $\mathcal{O}^G_{a_0}$ is a Toda orbit as defined in §1, $\mathcal{U}$ a neighborhood as before. Then the map

$$\mathcal{U} \cap \mathcal{O}^G_{a_0} \cap \mathcal{O}^L_{a_0} \to \Omega_{a_0}$$

given by

$$a \mapsto \{r^2_\mu(a) \ : \ \mu \in \Phi\}$$

is injective. The matrix entries of the inverse map depend rationally on the r^2_μ, with coefficients rational in the matrix entries of $P_\mu(a_0)$, $\mu \in \Phi$.

Proof: For $a \in \mathcal{U} \cap \mathcal{O}^G_{a_0} \cap \mathcal{O}^L_{a_0}$, take $X \in G$, $Y \in H$ so that

$$a_0 = \mathrm{Ad}\, X(a) = \mathrm{Ad}\, Y(a)$$

as assured by Theorem 4.5.

From the discussion preceding the proof of Theorem 6.9, it is clear that for suitable $\delta > 0$, $\bar{X} = \delta X$ has the properties (i) and (ii) of that theorem, with $\bar{X}_{\zeta\zeta} = 1$. Since $X^{-1}Y$ centralizes a, there will be positive $\bar{\kappa}_\mu$ so that

$$\bar{X}^{-1}Y = \sum_{\mu\in\Phi} {}_\mu P_\mu(a)$$

(in fact, $\bar{\kappa} = \delta^{-1}\kappa$, in the previous notation). It follows from the discussion preceding equation (G-L) that $\bar{X}$ solves

$$\bar{X}\bar{X}^{\dagger} = \sum_{\mu\in\Phi} \bar{\kappa}^{-2}_\mu P_\mu(a_0) \quad .$$

Hence (Theorem 6.9) $\bar{X}$ is the unique solution of this equation, and its matrix elements depend rationally on the $\bar{\kappa}_\mu$ and the matrix elements of $P_\mu(a_0)$. However, you see by using $\bar{X}$ instead of X in the discussion preceding the statement of this theorem that

$$\bar{\kappa}_\mu^{-2} r_\mu^2(a_0) = \langle v_0, \bar{X}^+ v_0\rangle^2 \; r_\mu^2(a)$$

$$= r_\mu^2(a)$$

so that

$$\bar{\kappa}_\mu = \frac{r_\mu(a)}{r_\mu(a_0)} \quad .$$

Since $\{\bar{\kappa}_\mu\}$ is determined uniquely by $\{r_\mu(a)\}$, and $\bar{X}$ by $\{\bar{\kappa}_\mu\}$, it follows that $a = \bar{X}\ a_0\bar{X}$ is determined uniquely by $\{r_\mu(a)\}$, and that the matrix elements are rational as advertised.

q.e.d.

__Remark__. To recover X from $\{r_\mu(a)\}$, observe that, since $\underline{\ell}$ is semisimple and V is a nontrivial and irreducible $\underline{\ell}$-module, any connected (and so any exponential) subgroup of L is represented by matrices of determinant 1. Therefore, since $\bar{X} = \delta X$

$$\det \bar{X} = \delta$$

but

$$\det \bar{X}\bar{X}^+ = (\det \bar{X})^2$$

$$= \prod_{\mu\in\Phi} \bar{\kappa}_\mu^{-2m_\mu}$$

where m_μ is the multiplicity of the weight μ in V. So

$$\delta = \prod_{\mu\in\Phi} \left(\frac{r_\mu(a_0)}{r_\mu(a)}\right)^{m_\mu}$$

which shows that even the matrix elements of $X \in G$ are rational in the r_μ and the matrix elements of $P_\mu(a_0)$.

__Remark__. From linear algebra one knows that the projections $P_\mu(a_0)$ are rational functions of the eigenvalues $\mu^*(a_0)$ and the symmetric matrix a_0. Therefore, the result may be rephrased: the matrix elements of a are rational in the $r_\mu^2(a)$, $\mu^*(a) = \mu^*(a_0)$, and the matrix entries of a_0.

The main lack of this set of results is that the set Ω is not a priori characterized. Nonetheless, you now have sufficient machinery to solve the equations of motion on Toda orbits arising from Ad-invariant Hamiltonians. The procedure is like this:

1. Given the initial condition a_0, compute the eigenvalues μ^*, and the projections $P_\mu(a_0)$, hence the numbers $r_\mu^2(a_0) = \langle v_0, P_\mu(a_0)v_0\rangle$.
2. Use Cor. 6.5 to compute the values of r_μ^2 along the trajectory.
3. Use the proof of Theorem 6.10 to compute the matrix elements along the trajectory from the result of step 2. Of course the values $\{r_\mu^2\}$ along the trajectory a priori belong to Ω_{a_0}, so the question of characterizing that set does not arise here.

The remarkable result is that the matrix entries of the trajectory are rational functions of linear exponentials, the initial conditions, and the (constant) eigenvalues. Thus the classical program outline in §1 has been carried out for all systems of Toda type.

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WS/jvs

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #1957	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) On Systems of Toda Type . 6		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) W. Symes 10		8. CONTRACT OR GRANT NUMBER(s) DAAG29-75-C-0024 MCS78-09525
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18.		12. REPORT DATE 11 May 1979
		13. NUMBER OF PAGES 65
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 12 69 p. 14 MRC-TSR-1957		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited. 9 Technical summary rept.,

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

15 DAAG29-75-C-0024, NSF-MCS78-09525

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Toda lattice, integrals in involution, inverse spectral problem, representation theory of semisimple Lie algebras.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report presents a group-theoretical interpretation of the structure of certain completely integrable Hamiltonian systems. These systems generalize the nonperiodic Toda Lattice system. Since all of them owe their special properties to the same group-theoretical mechanism, we call them systems of Toda type. This mechanism also provides a foundation for understanding many other completely integrable systems, including the Korteweg-de Vries equations, which will not be discussed here. We give an invariant definition of the class of systems of Toda type, and a method for explicit integration of all systems in this class in terms of rational combinations of linear exponential functions

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221 200